Tropical Toric Connections, Weierstrass Sets and Linear Degenerate Tropical Flag Varieties

by

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Contents

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself, except where stated otherwise, and it has not been submitted in any previous application for any degree or award. Chapter 1 and Chapter 2 have been carried out by the author only and are an adapted version of [14] and [15] respectively, Chapter 3 is an adapted version of [17] and is a collaboration with Victoria Schleis.

Abstract

Tropical geometry is a branch of mathematics between algebraic geometry, polyhedral geometry and combinatorics. One of the main ideas of the field is to associate to an algebraic variety X a tropical variety $\mathrm{trop}(X)$ called its *tropicalization.* This object is a polyhedral complex, and thus can be studied by means of polyhedral geometry and combinatorics, providing some information about the original variety X while being easier to study.

In Chapter 1, we study a connection between tropical geometry and intersection theory of toric varieties, providing an algorithm to compute toric intersection classes from tropical varieties. We outline an application to wonderful compactifications, in particular the effective cone of $\overline{M}_{0,n}$.

In Chapter 2, we study some possible tropical analogues of Weierstrass semigroups of an algebraic curve. We define two candidates, called the rank and functional Weierstrass sets, we study their interplay and properties.

In Chapter 3, we study a generalization of flag varieties, namely linear degenerate flag varieties, from a tropical point of view. We define a linear degenerate flag Dressian and prove a result analogous to the flag Dressian.

Introduction

Tropical geometry is a relatively new area of mathematics at the intersection of algebraic geometry, polyhedral geometry and combinatorics. In the last two decades, we have seen a rapid and significant development of this branch of mathematics [85]. One of the main ideas of the field is to associate to an algebraic variety X a tropical variety $\mathrm{trop}(X)$ called its *tropicalization*. This object is a polyhedral complex, and thus can be studied by means of polyhedral geometry and combinatorics, while being easier to study it still provides valuable information about the original variety X. This is the reason why tropical geometry has been used to study problems in algebraic geometry [86, 52, 51, 53] and combinatorics [62, 63, 1]. In addition, the introduction of this new geometry also gives rise to many interesting problems that are purely tropical [83, 84, 34, 103, 3].

This thesis is divided into three chapters, and the contents of each chapter are independent to each other.

In Chapter 1, we study an interesting connection between tropical geometry and toric intersection theory. The intersection theory of toric varieties was first studied in Fulton and Sturmfels [48], and it has many applications in different contexts, which include: wonderful and tropical compactifications [108, 31, 42], birational geometry [56, 57, 21], tropical intersection theory [75, 73, 93, 3, 74, 103] and combinatorial Hodge theory [62, 63, 1]. In a certain way, intersection classes of a toric variety with fan Σ can be thought in terms of balanced subfans of Σ, also referred as *Minkowski weights*. From the Structure Theorem of Tropical Geometry we know that the tropicalization of a subvariety of a torus $Y \subseteq T^n$ with trivial valuation is a balanced fan. A surprising connection between tropical and toric geometry is that the tropicalization of Y is the balanced fan corresponding to the intersection class of the closure of Y inside an "enough refined" toric variety. This fact provides us an algorithm, that we describe in Section 1.2.4, to compute toric intersection classes starting from the data of the tropicalization. This algorithm was implemented in a new package `TropicalToric.m2` for Macaulay2 [58] that we describe in Section 1.4. Further, we present some applications to the intersection theory of wonderful compactifications and the moduli space $\overline{M}_{0,n}$.

Chapter 2 is a readapted version of the work [15], which is about possible tropical analogues of the notion of Weierstrass semigroup of an algebraic curve. Two candidates are studied, called the rank and functional Weierstrass sets. These are defined within the framework of the Riemann-Roch theory on graphs developed by Baker and Norine [7]. We refer to Section 2.1 for a more thorough introduction.

Chapter 3 is a readapted version of the work [17], which is about linear degenerations of tropical flag varieties. This work studies linear degenerate flag varieties, which are a generalization of flags of linear spaces, from a tropical point of view. The notion of linear degenerate flag Dressian is introduced, and a characterization of its points, analogous to the flag Dressian, is proved. We refer to Section 3.1 for a more thorough introduction.

Chapter 1

Tropical methods in toric intersection theory

1.1 Intersection theory

1.1.1 Chow ring

Let X be a variety of dimension n. The group of k**-cycles** $Z_k(X) = Z^{n-k}(X)$ is the free abelian group generated by the irreducible subvarieties of X of dimension k. In other words, a k-cycle $V \in Z_k(X)$ is a formal linear combination with coefficients in $\mathbb{Z}$ of irreducible k-dimensional subvarieties V_i of X:

$$V = \sum_i a_i V_i \in Z_k(X).$$

With the notation above, we say that V is **effective** if $a_i \geq 0$ for every i. The cycles arising from an irreducible subvariety (i.e. the cycles V_i above), are sometimes called **prime** (or **irreducible**) cycles.

An $(n-1)$-cycle is thus a Weil divisor on X. If V is an irreducible subvariety of X of dimension $k+1$, and f is a rational function of V, then we can regard the principal divisor $\operatorname{div}(f)$ of V as a k-cycle of X. Denote by $\operatorname{Rat}_k(X) = \operatorname{Rat}^{n-k}(X)$ the subgroup of $Z_k(X)$ generated by all the k-cycles of the form $\operatorname{div}(f)$ for f a rational function of some irreducible subvariety of X of dimension $k+1$. The k-cycles in $\operatorname{Rat}_k(X)$ are called **principal** and two k-cycles are **rationally equivalent** if their difference is principal. The k-dimensional **Chow group** of X is the group of k-cycles modulo rational equivalence:

$$A_k(X) = Z_k(X)/\operatorname{Rat}_k(X).$$

The codimension k Chow group is defined by $A^k(X) = A_{n-k}(X)$.

If in addition X is smooth, then with a substantial amount of work (see [47, Chapter 8]) it can be shown that there is an intersection product $A^k(X) \times A^l(X) \to A^{k+l}(X)$ which coincides with the geometric intersection of cycles in the case of transverse intersections.

Proposition-Definition 1.1.1 ([47, Proposition 8.3]). *The intersection product makes*

$$A^*(X) = \bigoplus_{k=0}^{n} A^k(X)$$

into a commutative graded ring, called the ***Chow ring*** *of X.*

1.1.2 Effective and nef cones

In this section, X is a smooth projective variety of dimension n. For a cycle $V \in Z(X) = \bigoplus_k Z^k(X)$, denote by $[V]$ its class in the Chow ring $A^*(X)$. Note that since X is complete, there is a degree homomorphism $\deg : A^n(X) \to \mathbb{Z}$ defined by $\sum a_i V_i \mapsto \sum a_i$.

Definition 1.1.2 (Numerical equivalence). Two k-cycles $V_1, V_2 \in Z_k(X)$ are **numerically equivalent**, written $V_1 \equiv V_2$, if

$$\deg([V_1] \cdot [V]) = \deg([V_2] \cdot [V]) \quad \text{for every } V \in Z^k(X).$$

A k-cycle is **numerically trivial** if it is numerically equivalent to zero. The subgroup of numerically trivial k-cycles is denoted by $\mathrm{Num}_k(X) = \mathrm{Num}^{n-k}(X) \subseteq Z_k(X)$.

We denote the corresponding quotient groups by

$$N_k(X) = N^{n-k}(X) = Z_k(X)/\,\mathrm{Num}_k(X).$$

Definition 1.1.3 (Nef cycles). A k-cycle $V \in Z_k(X)$ is **nef** (or **numerically effective**) if

$$\deg([V] \cdot [V']) \geq 0 \quad \text{for all } V' \in Z^k(X) \text{ effective.}$$

Analogously, a Chow (or numerical) class α in $A_k(X)$ (or $N_k(X)$) is **effective** (or **nef**) if it is the class of an effective (or nef) cycle.

Note that, by definition, if two k-cycles $V_1, V_2 \in Z_k(X)$ are rationally equivalent $V_1 \sim V_2$, then they are also numerically equivalent $V_1 \equiv V_2$. In other words, we have $\mathrm{Rat}_k(X) \subseteq \mathrm{Num}_k(X)$, therefore $N_k(X)$ is a quotient of $A_k(X)$. This implies that the intersection product in $A^*(X)$, induces an intersection product on $N(X) = \bigoplus_k N^k(X)$.

Set $N_k(X)_{\mathbb{R}} = N_k(X) \otimes_{\mathbb{Z}} \mathbb{R}$. From [80, Proposition 1.1.16], $N_k(X)_{\mathbb{R}}$ is a vector space of finite dimension. Thus, the intersection product gives a non-degenerate bilinear pairing

$$N^k(X)_{\mathbb{R}} \times N_k(X)_{\mathbb{R}} \to \mathbb{R}.$$

For a k-cycle $C \in Z_k(X)$, denote by $[C]_n$ its class in $N_k(X)$.

Definition 1.1.4 (Effective and nef cones)**.**

- The k-th **nef cone** $\mathrm{Nef}_k(X)$ is the cone in $N_k(X)_{\mathbb{R}}$ generated by classes of nef k-cycles:

$$\mathrm{Nef}_k(X) = \mathrm{Nef}^{n-k}(X) = \left\{ \sum_{i=1}^{m} a_i [C_i]_n : a_i \geq 0,\ C_i \in Z_k(X) \text{ is nef} \right\} \subseteq N_k(X)_{\mathbb{R}}.$$

- The k-th **(pseudo)effective cone** $\mathrm{Eff}_k(X)$ is the closure of the cone in $N_k(X)_{\mathbb{R}}$ generated by classes of effective k-cycles:

$$\mathrm{Eff}_k(X) = \mathrm{Eff}^{n-k}(X) = \overline{\left\{ \sum_{i=1}^{m} a_i [C_i]_n : a_i \geq 0,\ C_i \in Z_k(X) \text{ is effective} \right\}} \subseteq N_k(X)_{\mathbb{R}}.$$

Sometimes the notation $\overline{\mathrm{Eff}_k}(X)$ is used to distinguish the cone generated by effective classes from its closure. We will not use this notation, and denote simply by $\mathrm{Eff}_k(X)$ the closure of the cone generated by the effective classes.

The $(n-1)$-st nef cone often times is simply called nef cone, and is usually denoted by $\mathrm{Nef}(X) = \mathrm{Nef}^1(X)$. The first effective cone is called the effective cone of curves, and it is denoted by $\overline{\mathrm{NE}}(X) = \mathrm{Eff}_1(X)$. The following result easily follows from the definitions.

Corollary 1.1.5. *The cones $\mathrm{Nef}^k(X)$ and the $\mathrm{Eff}_k(X)$ are dual to each other with respect to the intersection product.*

1.2 Tropical Toric connections

1.2.1 Toric intersection theory

In this section we recall the basics of toric intersection theory. All toric varieties throughout are assumed to be normal.

We will follow the usual notation adopted in toric geometry. A toric variety X_Σ is defined by a rational polyhedral fan Σ in $N_{\mathbb{R}} = N \otimes \mathbb{R}^n$ for a lattice $N \simeq \mathbb{Z}^n$, where $n = \dim X_\Sigma$. The lattice dual to N is denoted by $M = \mathrm{Hom}(N, \mathbb{Z})$, and set $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

We denote by $\Sigma(k)$ the set of k-dimensional cones of Σ. The support of Σ is denoted by $|\Sigma|$. The algebraic torus of X_Σ is denoted by $T \simeq (K^*)^n$, where K is the base field of X_Σ. Each cone $\sigma \in \Sigma$ determines an orbit $O(\sigma)$ of the action of the torus T on X_Σ. Its closure $V(\sigma) = \overline{O(\sigma)}$ is a union of torus orbits and it is a torus invariant irreducible subvariety of X_Σ of dimension $n - \dim(\sigma)$.

Proposition 1.2.1 ([46, Section 5.1][28, Lemma 12.5.1])**.** *Let X_Σ be a toric variety. The Chow group $A^k(X_\Sigma)$ is generated by the classes*

$$\{[V(\sigma)] : \sigma \in \Sigma(k)\}.$$

Now we give a combinatorial description of the relations of the set of generators above. For every cone $\sigma \in \Sigma$ let N_σ be the sublattice of N generated by $\sigma \cap N$, let $N(\sigma) = N/N_\sigma$ and $M(\sigma) = \sigma^\perp \cap M$ the dual lattice of $N(\sigma)$. Now fix $\tau \in \Sigma(k-1)$, for every $\sigma \in \Sigma(k)$ such that $\tau \subseteq \sigma$, the variety $V(\sigma)$ is a codimension 1 subvariety of $V(\tau)$, and thus defines a divisor on $V(\tau)$. Let $n_{\sigma,\tau}$ be a lattice point in σ whose image generates the one dimensional lattice N_σ/N_τ. Any $u \in M(\tau)$ determines a rational function x^u on $V(\tau)$ whose divisor is

$$\operatorname{div}(x^u) = \sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subseteq \sigma}} \langle u, n_{\sigma,\tau} \rangle V(\sigma)$$

therefore, in $A^k(X_\Sigma)$ we have

$$\sum_{\substack{\sigma \in \Sigma(k) \\ \tau \subseteq \sigma}} \langle u, n_{\sigma,\tau} \rangle [V(\sigma)] = 0. \tag{1.1}$$

Proposition 1.2.2 ([48, Proposition 2.1])**.** *The group of relations on the set of generators $\{[V(\sigma)] : \sigma \in \Sigma(k)\}$ of the Chow group $A^k(X_\Sigma)$ is generated by relations of the form* (1.1) *where τ ranges over $\Sigma(k-1)$.*

The Chow ring is well defined if X_Σ is smooth. However, for toric varieties, if Σ is just simplicial, we can define the intersection product on rational cycles, making

$$A^*(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{k=0}^{n} \left(A^k(X_\Sigma) \otimes_{\mathbb{Z}} \mathbb{Q} \right)$$

into a graded ring (see [46, Section 5] or [75, Section 2]). We now give an explicit description of this intersection product. For every cone $\sigma \in \Sigma(k)$ generated by the rays $\rho_1, \ldots, \rho_k \in \Sigma(1)$, define the **multiplicity** $\operatorname{mult}(\sigma)$ of σ as the index of the sublattice generated by $v_1, \ldots, v_k$, where v_i is the first lattice point of ρ_i. Explicitly, the index of such sublattice of $N \simeq \mathbb{Z}^n$ is the GCD of the maximal minors of the matrix with columns the v_i's seen as vectors in $\mathbb{Z}^n$.

Proposition 1.2.3 ([46, Section 5][28, Lemma 12.5.2])**.** *Let X_Σ be a simplicial toric variety. If $\rho_1, \ldots, \rho_k \in \Sigma(1)$ are distinct, then in $A^*(X_\Sigma)$ we have*

$$[V(\rho_1)] \cdot [V(\rho_2)] \cdot \cdots \cdot [V(\rho_k)] = \begin{cases} \frac{1}{\mathrm{mult}(\sigma)}[V(\sigma)] & \text{if } \sigma = \rho_1 + \cdots + \rho_k \in \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if X_Σ is smooth, the class of $V(\sigma)$, for $\sigma = \rho_1 + \cdots + \rho_k \in \Sigma(k)$ with $\rho_i \in \Sigma(1)$, is the intersection of the classes of $V(\rho_1), \ldots, V(\rho_k)$.

Now we want to explicitly describe the Chow ring of a complete simplicial toric variety X_Σ. Suppose that Σ has s rays $\rho_1, \ldots, \rho_s$, and let $D_1, \ldots, D_s$ be the corresponding torus invariant divisors. From Proposition 1.2.3, we have that the Chow ring is generated, as a graded ring, by the classes $[D_1], \ldots, [D_s]$ of such divisors. A first set of relations between these classes comes from Proposition 1.2.2. More precisely, if $V = (v_{ij})$ is the $n \times s$ matrix with columns the first lattice points v_i of the rays ρ_i of Σ, these relations are encoded in the ideal:

$$L(\Sigma) = \left\langle \sum_{j=1}^{s} v_{ij} D_j : i \in \{1, \ldots, n\} \right\rangle \subseteq \mathbb{Z}[D_1, \ldots, D_s].$$

The second set of relations comes from Proposition 1.2.3. In fact, if $I \subseteq \{1, \ldots, s\}$ is such that $\sum_{i \in I} \rho_i \notin \Sigma$, then $\prod_{i \in I} [D_i] = 0$, where here the sum of cones is intended as a *Minkowski sum*. These relations are encoded in the following ideal:

$$\mathrm{SR}(\Sigma) = \left\langle \prod_{i \in I} D_i : I \subseteq \{1, \ldots, s\}, \sum_{i \in I} \rho_i \notin \Sigma \right\rangle \subseteq \mathbb{Z}[D_1, \ldots, D_s].$$

The ideal $\mathrm{SR}(\Sigma)$ is the so-called *Stanley-Reisner* ideal (see [88, Definition 1.6]) of the following simplicial complex associated to Σ:

$$\Delta = \left\{ I \subseteq \{1, \ldots, s\} : \sum_{i \in I} \rho_i \in \Sigma \right\}.$$

Theorem 1.2.4 ([46, Section 5.2][28, Theorem 12.5.3][85, Theorem 6.7.1])**.** *Let X_Σ be a complete smooth toric variety whose fan Σ has s rays, with torus invariant divisors $D_1, \ldots, D_s$. The Chow ring of X_Σ is given by*

$$A^*(X_\Sigma) \simeq \mathbb{Z}[D_1, \ldots, D_s]/(L(\Sigma) + \mathrm{SR}(\Sigma)).$$

This holds with $\mathbb{Z}$ replaced by $\mathbb{Q}$ when X_Σ is simplicial.

Now assume that X_Σ is complete and simplicial. Since X_Σ is complete, there is a degree homomorphism $\deg : A^n(X_\Sigma) \to \mathbb{Z}$. For every $0 \leq k \leq n$, this homomorphism gives rise to another homomorphism

$$\mathscr{D}^k_{X_\Sigma} : A^k(X_\Sigma) \to \mathrm{Hom}(A^{n-k}(X_\Sigma), \mathbb{Z})$$

defined by $\mathscr{D}^k_{X_\Sigma}(\alpha)(\beta) = \deg(\alpha \cdot \beta)$ for every $\alpha \in A^k(X_\Sigma)$ and $\beta \in A^{n-k}(X_\Sigma)$.

Proposition 1.2.5 (Kronecker duality [48, Proposition 2.4])**.** *If X_Σ is a complete simplicial toric variety, then $\mathscr{D}^k_{X_\Sigma}$ is an isomorphism for every $0 \leq k \leq n$.*

The previous result gives us the following alternative description of Chow groups of a simplicial complete toric variety in terms of balanced fans.

Definition 1.2.6. A **weighted fan** is a pair (Σ, m) of a fan Σ pure of dimension d, and a weight function $m : \Sigma(d) \to \mathbb{Z}$. A weighted fan (Σ, m) is **balanced at** $\tau \in \Sigma(d-1)$ if

$$\sum_{\sigma \in \Sigma(d),\, \tau \subseteq \sigma} m(\sigma) n_{\sigma,\tau} = 0.$$

The weighted fan (Σ, m) is **balanced** if it is balanced at every $\tau \in \Sigma(d-1)$.

Balanced fans in [48] are called **Minkowski weights**. If Σ is a complete fan of dimension n, then for every $0 \leq k \leq n$ the Minkowski weights of the form $(\Sigma(k), m)$ with the operation $(\Sigma(k), m) + (\Sigma(k), m') = (\Sigma(k), m + m')$ form an abelian group. By combining Proposition 1.2.2 with Proposition 1.2.5 we have the following result.

Corollary 1.2.7 ([48, Theorem 3.1])**.** *If X_Σ is complete and simplicial, the Chow group $A^k(X_\Sigma)$ is isomorphic to the group of Minkowski weights of codimension k in Σ.*

1.2.2 Tropical compactifications

In this thesis, when we talk about tropicalization, we will always refer to *embedded tropicalization* in the sense of [85, Chapter 3].

Fix a field K with trivial valuation. A subvariety of an algebraic torus $T = (K^*)^n$ is called **very affine**. Throughout this section, if not specified otherwise, we will consider the following situation: $Y \subseteq T$ is a very affine variety irreducible of dimension d, X_Σ is a toric variety with torus T, and $\overline{Y}$ is the closure of Y inside X_Σ. In this situation, from the Structure Theorem of Tropical Geometry [85, Theorem 3.3.5], $\mathrm{trop}(Y)$ is the support of a balanced fan in $\mathbb{R}^n$. Note that Σ and $\mathrm{trop}(Y)$ have the same ambient space.

Theorem 1.2.8 ([85, Theorem 6.3.4][108, Lemma 2.2]). *For any $\sigma \in \Sigma$, $\overline{Y}$ intersects the torus orbit $O(\sigma)$ if and only if* $\operatorname{trop}(Y)$ *intersects the relative interior of σ.*

Corollary 1.2.9 ([85, Proposition 6.4.7][108, Proposition 2.3]). *The closure $\overline{Y}$ is complete if and only if* $\operatorname{trop}(Y) \subseteq |\Sigma|$.

Definition 1.2.10. Let $Y \subseteq T$ be a very affine variety. A **tropical compactification** of Y is any variety isomorphic to the closure $\overline{Y}$ in a toric variety X_Σ with $\operatorname{trop}(Y) = |\Sigma|$. If in addition the multiplication map $\psi : T \times \overline{Y} \to X_\Sigma$ given by $(t, x) \mapsto tx$ is flat and surjective, the tropical compactification is **flat tropical**.

Proposition 1.2.11 ([85, Proposition 6.4.14][108, Proposition 2.5]). *If the closure $\overline{Y}$ in X_Σ is a flat tropical compactification, then every refinement of Σ has this property. In addition* $\operatorname{trop}(Y) = |\Sigma|$.

Proposition 1.2.12 ([85, Proposition 6.4.15]). *Suppose that the closure $\overline{Y}$ in X_Σ is a flat tropical compactification, and that X_Σ is smooth. Then $\overline{Y}$ is Cohen-Macaulay at every point $p \in \overline{Y} \cap O(\sigma)$ for every $\sigma \in \Sigma(d)$.*

Theorem 1.2.13 ([108, Theorem 1.2]). *Any very affine variety $Y \subseteq T$ has a flat tropical compactification $\overline{Y}$ such that the corresponding toric variety X_Σ is smooth.*

For an explicit construction of a flat tropical compactification of a very affine variety, see [85, Proposition 6.4.17].

1.2.3 Toric intersection classes from tropical varieties

Throughout this section and the rest of this thesis, by intersection class of a cycle we mean its class in the Chow ring. We have seen in Section 1.2.1 that intersection classes of a complete simplicial toric variety X_Σ correspond to balanced fans. Let $Y \subseteq T$ be an irreducible very affine variety of dimension d such that its closure $\overline{Y}$ in X_Σ is a tropical compactification. We can give $\operatorname{trop}(Y)$ the structure of a balanced fan.

Question 1.2.14. Which intersection class corresponds to the balanced fan $\operatorname{trop}(Y)$?

In this section, we will prove that the answer to the previous question is the intersection class $[\overline{Y}] \in A_d(X_\Sigma)$. This fact has many interesting applications. For instance, intersection classes of toric varieties can be recovered using tropical geometry.

In this section and throughout, all degree homomorphisms of complete toric varieties will be denoted by deg.

Theorem 1.2.15 ([85, Theorem 6.7.7]). *Let $Y \subseteq T$ be a subvariety and let $\overline{Y}$ be a flat tropical compactification of Y in a smooth toric variety X_Σ. Let Σ' be a smooth completion of the fan Σ and let $i : X_\Sigma \to X_{\Sigma'}$ be the induced inclusion. Then, for every maximal cone σ in Σ we have*

$$m(\sigma) = \deg\left(i_*\left([\overline{Y}]\right) \cdot [V(\sigma)]\right),$$

where $m(\sigma)$ is the multiplicity of σ in $\operatorname{trop}(Y)$.

Proof. Let $d = \dim(\overline{Y})$. Since $\overline{Y}$ is a flat tropical compactification, from Proposition 1.2.11 we have $\operatorname{trop}(Y) = |\Sigma|$. Fix a maximal cone $\sigma \in \Sigma(d)$. By [85, Proposition 6.4.7 (2)], the scheme $\overline{Y} \cap O(\sigma)$ is zero dimensional. Since X_Σ is smooth, by Proposition 1.2.12, $\overline{Y}$ is Cohen-Macaulay at any point $p \in \overline{Y} \cap O(\sigma)$. We have

$$i_*\left([\overline{Y}]\right) \cdot [V(\sigma)] = \sum_{p \in \overline{Y} \cap O(\sigma)} i(p, \overline{Y} \cdot V(\sigma); X_{\Sigma'})[p] \in A_0(X_{\Sigma'}),$$

where $i(p, \overline{Y} \cdot V(\sigma); X_{\Sigma'})$ is the intersection multiplicity of p in $\overline{Y} \cdot V(\sigma)$ (see [47, Definition 7.1]). By [47, Proposition 7.1], $\sum_p i(p, \overline{Y} \cdot V(\sigma); X_{\Sigma'})$ equals the length of $\overline{Y} \cap O(\sigma)$. By [85, Remark 6.4.18], $\overline{Y} \cap O(\sigma)$ equals the quotient of the subscheme T defined by the initial ideal $\operatorname{in}_w(I_Y)$ by the torus $T_\sigma = N_\sigma \otimes K^*$, where I_Y is the Laurent ideal of $Y \subseteq T$. Since K has the trivial valuation, K equals its residue field. Thus by [85, Lemma 3.4.7] the length equals $m(\sigma)$. □

Now, we want to generalize the previous result to tropical compactifications that are not necessarily flat, with toric variety X_Σ simplicial.

Remark 1.2.16. Let Y be an irreducible d-dimensional subvariety of T and let $\overline{Y}$ be the closure of Y in a toric variety X_Σ such that $\operatorname{trop}(Y) = |\Sigma|$. Let $Z^d(X_\Sigma)$ be the group of codimension-d cycles of X_Σ, and $\operatorname{Rat}^d(X_\Sigma)$ the subgroup of principal codimension-d cycles. Now consider the homomorphism of groups $\varphi : Z^d(X_\Sigma) \to \mathbb{Z}$ defined by $\varphi(V(\sigma)) = m(\sigma)$, where $m(\sigma)$ is the multiplicity of σ in $\operatorname{trop}(Y)$. By using the same notation as Section 1.2.1, since $(\operatorname{trop}(Y), m)$ is a balanced fan, for every $\tau \in \Sigma(d-1)$ we have

$$\sum_{\substack{\tau \subseteq \sigma \\ \sigma \in \Sigma(d)}} n_{\sigma,\tau} m(\sigma) = 0.$$

Thus, for every $u \in M(\tau)$ it follows

$$\varphi\left(\sum_{\substack{\tau \subseteq \sigma \\ \sigma \in \Sigma(d)}} \langle u, n_{\sigma,\tau}\rangle V(\sigma)\right) = \sum_{\substack{\tau \subseteq \sigma \\ \sigma \in \Sigma(d)}} \langle u, n_{\sigma,\tau}\rangle m(\sigma) = \left\langle u, \sum_{\substack{\tau \subseteq \sigma \\ \sigma \in \Sigma(d)}} n_{\sigma,\tau} m(\sigma)\right\rangle = 0.$$

From Proposition 1.2.2, this means that $\mathrm{Rat}^d(X_\Sigma) \subseteq \ker\varphi$, therefore φ induces a well defined homomorphism $\varphi' : A^d(X_\Sigma) \to \mathbb{Z}$ with $\varphi'\big([V(\sigma)]\big) = m(\sigma)$.

Let Σ be a fan. A **subfan** of Σ is a subset $\Sigma' \subseteq \Sigma$ such that Σ' is also a fan. In other words, every cone in Σ' is also a cone in Σ. The next result appears in various versions in the literature, see for instance [75, Lemma 2.3] or [73, Section 9]. We provide here a proof using just Theorem 1.2.15.

Theorem 1.2.17. *Let $Y \subseteq T$ be an irreducible subvariety and let $\overline{Y}$ be the closure of Y in a simplicial toric variety X_Σ such that* $\mathrm{trop}(Y) = |\Sigma|$. *Let Σ' be a simplicial completion of the fan Σ and let $i : X_\Sigma \to X_{\Sigma'}$ be the induced inclusion. Then, for every maximal cone σ in Σ we have*

$$m(\sigma) = \deg\big(i_*\big([\overline{Y}]\big)\cdot[V(\sigma)]\big),$$

where $m(\sigma)$ is the multiplicity of σ in $\mathrm{trop}(Y)$.

Proof. Let $d = \dim(\overline{Y})$. Take a smooth refinement $\tilde{\Sigma}'$ of Σ' that has a subfan $\tilde{\Sigma}$ with the following properties:

(1) the closure $\overline{Y'}$ of Y in $X_{\tilde{\Sigma}}$ is a flat tropical compactification of Y;

(2) $\tilde{\Sigma}$ is a refinement of Σ.

This is possible since from Theorem 1.2.13 every very affine variety has a flat tropical compactification; and from Proposition 1.2.11 if a fan gives rise to a flat tropical compactification, then so does any of its refinements.

The fans we are considering are summarized in the following diagram:

$$\begin{tikzcd}
\tilde{\Sigma} \arrow[r, "\text{completion}"] & \tilde{\Sigma}' \\
\Sigma \arrow[u, "\text{refinement}"] \arrow[r, "\text{completion}"] & \Sigma' \arrow[u, "\text{refinement}"']
\end{tikzcd}$$

Let $\pi : X_{\tilde{\Sigma}'} \to X_{\Sigma'}$ be the induced toric map and denote by i both the inclusion maps of $X_\Sigma \subseteq X_{\Sigma'}$ and $X_{\tilde{\Sigma}} \subseteq X_{\tilde{\Sigma}'}$. Consider the following diagram of homomorphisms

$$\begin{tikzcd}
A^d(X_{\tilde{\Sigma}'}) \arrow[rr, "\psi"] \arrow[dr, "\pi_*"'] & & \mathbb{Z} \\
& A^d(X_{\Sigma'}) \arrow[ur, "\varphi"'] &
\end{tikzcd} \tag{1.2}$$

defined as follows. Fix $\sigma' \in \tilde{\Sigma}'(d)$ and $\sigma \in \Sigma'(d)$ such that $\pi(\sigma') \subseteq \sigma$. The map π_* is the pushforward homomorphism, thus $\pi_*\big([V(\sigma')]\big) = [V(\sigma)]$. The map ψ is defined by

$$\psi\big([V(\sigma')]\big) = \deg\big(i_*\big([\overline{Y'}]\big) \cdot [V(\sigma')]\big).$$

The map ψ is well defined since it is the composition of the homomorphism deg and the multiplication in $A^*(X_{\tilde{\Sigma}'})$ of the class $i_*\big([\overline{Y'}]\big)$. The map φ is defined by

$$\varphi\big([V(\sigma)]\big) = \begin{cases} m(\sigma) & \sigma \in \Sigma(d), \\ 0 & \text{otherwise.} \end{cases}$$

It is well defined since $(\mathrm{trop}(Y), m)$ is a balanced fan (see Remark 1.2.16).

Now the diagram (1.2) commutes, since the maps ψ and $\varphi \circ \pi_*$ are equal on the set of generators $\{[V(\sigma')] : \sigma' \in \tilde{\Sigma}'(d)\}$. In fact, let σ and σ' as above, if $\sigma' \in \tilde{\Sigma}(d)$, so we also have $\sigma \in \Sigma(d)$, then from Theorem 1.2.15 it follows $\psi\big([V(\sigma')]\big) = m(\sigma)$, therefore

$$\psi\big([V(\sigma')]\big) = m(\sigma) = \varphi\big([V(\sigma)]\big) = \varphi\big(\pi_*\big([V(\sigma')]\big)\big).$$

On the other hand, if $\sigma' \notin \tilde{\Sigma}(d)$, this means that the relative interior of any cone $\tau \in \tilde{\Sigma}'$ containing σ' does not intersect $\mathrm{trop}(Y)$. Hence, from Theorem 1.2.8, $\overline{Y'} \cap V(\sigma') = \emptyset$, so $\psi\big([V(\sigma')]\big) = 0$. Further, we also have $\sigma \notin \Sigma(d)$, thus $\varphi\big(\pi_*\big([V(\sigma')]\big)\big) = \varphi\big([V(\sigma)]\big) = 0$, so ψ and $\varphi \circ \pi_*$ are equal.

Finally, for every $\sigma \in \Sigma(d)$ we have:

$$\begin{aligned}
\deg\big(i_*\big([\overline{Y}]\big) \cdot [V(\sigma)]\big) &= \deg\Big(\pi_*\big(i_*\big([\overline{Y'}]\big)\big) \cdot [V(\sigma)]\Big) && (1.3)\\
&= \deg\Big(\pi_*\Big(i_*\big([\overline{Y'}]\big) \cdot \pi^*\big([V(\sigma)]\big)\Big)\Big) && (1.4)\\
&= \deg\big(i_*\big([\overline{Y'}]\big) \cdot \pi^*\big([V(\sigma)]\big)\big) && (1.5)\\
&= \psi\big(\pi^*\big([V(\sigma)]\big)\big) = \varphi\big(\pi_*\big(\pi^*\big([V(\sigma)]\big)\big) = \varphi\big([V(\sigma)]\big) = m(\sigma) && (1.6)
\end{aligned}$$

where: in (1.3) we used that $\pi_*\big(i_*\big([\overline{Y'}]\big)\big) = i_*\big([\overline{Y}]\big)$ since $\overline{Y}'$ is the strict transform of $\overline{Y}$ in $X_{\tilde{\Sigma}}$; in (1.4) we used the projection formula [47, Proposition 2.3 (c)]; in (1.5) we used the fact that the restriction of the pushforward $\pi_{*|A_0(X_{\tilde{\Sigma}'})} : A_0(X_{\tilde{\Sigma}'}) \to A_0(X_{\Sigma'})$ is an isomorphism; in (1.6) we used: the definition of ψ, that $\psi = \varphi \circ \pi_*$, the properties of the pushforward and pullback homomorphisms, and the definition of φ. $\square$

1.2.4 Tropical algorithm for toric intersection classes

Theorem 1.2.17 implicitly provides us an algorithm to compute the intersection class of a subvariety of a toric variety from the data of its tropicalization. In this section, we are going to describe this algorithm in detail.

Algorithm 1.2.18. The input is a complete simplicial toric variety X_Σ and the Laurent ideal $I \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the output is the class $[\overline{Y}] \in A^{n-d}(X_\Sigma)$, where $Y = V(I)$ with $d = \dim Y$:

1. Compute $\operatorname{trop}(V(I))$.

2. Compute a simplicial fan $\tilde{\Sigma}$ with support $\operatorname{trop}(V(I))$, and compute the multiplicities $m(\sigma')$ of $\sigma' \in \tilde{\Sigma}(d)$.

3. Choose a basis $\{[V(\sigma')] : \sigma' \in B \subseteq \tilde{\Sigma}(d)\}$ of $A^d(X_{\tilde{\Sigma}})$, let $\pi : X_{\tilde{\Sigma}} \to X_\Sigma$ be the induced toric map, and compute the pullbacks $\pi^*([V(\sigma)]) = \sum_{\sigma' \in B} a_{\sigma,\sigma'}[V(\sigma')]$ for every $\sigma \in \Sigma(d)$.

4. Compute the numbers $m(\sigma) = \sum_{\sigma' \in B} a_{\sigma,\sigma'} m(\sigma')$ for every $\sigma \in \Sigma(d)$.

5. Return the class $\alpha \in A^{n-d}(X_\Sigma)$ corresponding by Kronecker duality to the function in $\operatorname{Hom}(A^d(X_\Sigma), \mathbb{Z})$ defined by $[V(\sigma)] \mapsto m(\sigma)$ for every $\sigma \in \Sigma(d)$.

Remark 1.2.19. The above algorithm requires the toric variety to be complete. We need this hypothesis only in the last step in order to apply Kronecker duality. However, if the variety X_Σ is smooth (and not necessarily complete), then we could instead use Poincaré duality [47, Corollary 17.4]. On a practical level, even in this context, the above algorithm would be exactly the same.

We now outline a proof of correctness of the above algorithm. Let X_Σ be a complete simplicial toric variety of dimension n, and let $Z \subseteq X_\Sigma$ be a d-dimensional irreducible subvariety that is not contained in the toric boundary. In other words, suppose that $Z \cap T \neq \emptyset$. Set $Y = Z \cap T$. Then we have $\overline{Y} = Z$ since T is dense in X_Σ. If we fix coordinates on the torus $T \simeq (K^*)^n$, then we have $Y = V(I)$ for some ideal I in the Laurent ring $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. We now want to compute the class $[\overline{Y}] \in A^{n-d}(X_\Sigma)$ starting from the data of the tropicalization $\operatorname{trop}(Y) = \operatorname{trop}(V(I))$ and its multiplicities (in the sense of [85, Definition 3.4.3]). In order to do so, we first observe that it would be enough to compute the intersection numbers $\deg([\overline{Y}] \cdot [V(\sigma)])$ for every $\sigma \in \Sigma(d)$. In fact, as X_Σ is complete and simplicial, by Kronecker duality (Proposition 1.2.5) there is an isomorphism $A^{n-d}(X_\Sigma) \simeq \operatorname{Hom}(A^d(X_\Sigma), \mathbb{Z})$ that sends every class α to the function $\beta \mapsto \deg(\alpha \cdot \beta)$. Further the classes $[V(\sigma)]$ for $\sigma \in \Sigma(d)$ generate $A^d(X_\Sigma)$, therefore, the function corresponding via Kronecker duality to the class $[\overline{Y}]$ is determined by the intersection numbers $\deg([\overline{Y}] \cdot [V(\sigma)])$.

Now, note that in order to compute these intersection numbers, we cannot directly apply Theorem 1.2.17, since $\operatorname{trop}(Y)$ is not necessarily the support of a subfan of Σ. In order to do so, we can proceed as follows. Take a simplicial refinement Σ' of Σ such that it contains a subfan $\tilde{\Sigma}$ that has support $\operatorname{trop}(Y)$ (for instance, we could take the common refinement of Σ and a completion of a fan with support $\operatorname{trop}(Y)$). Note that Σ' is complete and simplicial. Let $\pi : X_{\Sigma'} \to X_\Sigma$ be the induced toric map. Let $\overline{Y}'$ be the strict transform of $\overline{Y}$ in $X_{\Sigma'}$. In other words, $\overline{Y}'$ is the closure of Y in $X_{\Sigma'}$. By construction, we have $\pi_*([\overline{Y}']) = [\overline{Y}]$. By the same argument used in the proof of Theorem 1.2.17 (specifically line (1.5)) we have

$$\deg\left([\overline{Y}] \cdot [V(\sigma)]\right) = \deg\left([\overline{Y}'] \cdot \pi^*([V(\sigma)])\right) \quad \text{for every } \sigma \in \Sigma(d).$$

Further, since the classes $[V(\sigma')]$ for $\sigma' \in \Sigma'(d)$ generate $A^d(X_{\Sigma'})$, by expressing every intersection class as the intersection product of divisors classes and by using [28, Theorem 4.2.12] and [28, Proposition 6.2.7], we can compute an expression of the pullbacks $\pi^*([V(\sigma)]) \in A^d(X_{\Sigma'})$ in terms of the classes $[V(\sigma')]$. Therefore, the intersection numbers $\deg\left([\overline{Y}'] \cdot \pi^*([V(\sigma)])\right)$ for $\sigma \in \Sigma(d)$ can be computed starting from the numbers $\deg\left([\overline{Y}'] \cdot [V(\sigma')]\right)$ for $\sigma' \in \Sigma'(d)$. Finally, we now can apply Theorem 1.2.17 on $\overline{Y}'$ and X'_Σ, so for every $\sigma' \in \Sigma'(d)$, we obtain

$$m(\sigma') = \deg\left([\overline{Y}'] \cdot ([V(\sigma')])\right)$$

where $m(\sigma')$ is the multiplicity of σ' in $\operatorname{trop}(Y)$ (with $m(\sigma) = 0$ if $\sigma \notin \tilde{\Sigma}$).

Remark 1.2.20. Note that it is not actually necessary to explicitly construct the fan Σ', but it is enough to compute $\tilde{\Sigma}$ and perform all the computations on $X_{\tilde{\Sigma}}$, as for all the cones $\sigma' \notin \tilde{\Sigma}$ we have $m(\sigma') = 0$.

1.3 An application to wonderful compactifications

1.3.1 Nested sets and Bergman fans

Following [42], we introduce the notions of building set and nested set complex on a lattice. Let $\mathcal{L}$ be a finite lattice and denote by $\hat{0}$ the least element. For any subset $\mathcal{G}$ of $\mathcal{L}$ we denote by $\max \mathcal{G}$ the set of maximal elements of $\mathcal{G}$. For any $X \in \mathcal{L}$ set $\mathcal{G}_{\leq X} = \{G \in \mathcal{G} : G \leq X\}$. We denote intervals by $[X, Y] = \{Z \in \mathcal{L} : X \leq Z \leq Y\}$, thus $\mathcal{G}_{\leq X} = [\hat{0}, X] \cap \mathcal{G}$.

Definition 1.3.1 (Building set)**.** Let $\mathcal{L}$ be a finite lattice. A subset $\mathcal{G}$ of $\mathcal{L} \setminus \{\hat{0}\}$ is a **building set** of $\mathcal{L}$ if for every $X \in \mathcal{L} \setminus \{\hat{0}\}$ and $\max \mathcal{G}_{\leq X} = \{G_1, \ldots, G_k\}$ there is an isomorphism of posets

$$\varphi_X : \prod_{i=1}^{k} [\hat{0}, G_i] \xrightarrow{\cong} [\hat{0}, X]$$

with $\varphi_X(\hat{0}, \ldots, G_i, \ldots, \hat{0}) = G_i$ for $i \in \{1, \ldots, k\}$. We call $\max \mathcal{G}_{\leq X}$ the **set of factors** of X in $\mathcal{G}$.

There are two extreme examples of building sets: the (inclusion) maximal building set $\mathcal{L} \setminus \hat{0}$, and the (inclusion) minimal building set, consisting of all elements $X \in \mathcal{L} \setminus \hat{0}$ for which $[\hat{0}, X]$ does not decompose as a direct product of posets, the so-called **irreducibles**.

Definition 1.3.2 (Nested set complex)**.** Let $\mathcal{L}$ be a finite lattice and $\mathcal{G}$ be a building set in $\mathcal{L}$. A subset $\mathcal{S}$ of $\mathcal{G}$ is **nested** if, for any subset $\{G_1, \ldots, G_t\} \subseteq \mathcal{S}$ of pairwise incomparable elements of cardinality at least two, the join $G_1 \vee \cdots \vee G_t$ does not belong to $\mathcal{G}$. The nested sets of $\mathcal{G}$ form an abstract simplicial complex $\mathcal{N}(\mathcal{L}, \mathcal{G})$, called the **nested set complex** of $\mathcal{G}$.

The nested set complex of the maximal building $\mathcal{L} \setminus \hat{0}$ is the order complex of $\mathcal{L} \setminus \hat{0}$: the abstract simplicial complex consisting of all the chains of $\mathcal{L} \setminus \hat{0}$.

We now define the Bergman fan of a matroid. We assume that the reader is familiar with the basics of matroid theory, for which our main reference will be [94]. Let M be a matroid on the ground set $[n] = \{0, 1, \ldots, n\}$. The flats of M ordered by inclusion form a finite lattice $\mathcal{L}(M)$, called the **lattice of flats** of M. For a subset $S \subseteq [n]$ we set $e_S = \sum_{i \in S} e_i \in \mathbb{R}^{n+1}$ and denote by $\mathbb{R}\mathbf{1}$ the subspace of $\mathbb{R}^{n+1}$ generated by $e_{[n]}$. For every subset $\mathcal{S} \subseteq \mathcal{L}(M)$, we denote by $\sigma_{\mathcal{S}}$ the cone in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ spanned by the classes of the vectors $\{e_S : S \in \mathcal{S}\}$.

Definition 1.3.3 (Bergman fan)**.** Let M be a matroid on the ground set $[n]$ of rank r, and let $\mathcal{G}$ be a building set of the lattice of flats $\mathcal{L}(M)$. The *Bergman fan* $B_{\mathcal{G}}(M)$ of M, with respect to the building set $\mathcal{G}$, is the $(r-1)$-dimensional fan in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ consisting of the cones $\{\sigma_{\mathcal{S}} : \mathcal{S} \in \mathcal{N}(\mathcal{L}, \mathcal{G})\}$.

The rays of the Bergman fan corresponds to the elements of $\mathcal{G}$. The support of the Bergman fan depends just on the matroid, whereas the fan structure is given by the building set. Bergman fans are the *tropical linear spaces* with the trivial valuation, as in this case the tropicalization of a linear space L is the Bergman fan of the (realizable) matroid $M(L)$ associated to L. In general, Bergman fans are pure dimensional, and if we assign to each maximal cone of a Bergman fan $B_{\mathcal{G}}(M)$ weight one, then we obtain a balanced fan. The following result follows from [42, Proposition 2]:

Proposition 1.3.4. *Any Bergman fan is smooth.*

1.3.2 Hyperplane arrangements

Let $\mathcal{A} = \{H_i : 0 \le i \le n\}$ be an arrangement of $n+1$ hyperplanes in $\mathbb{P}^d$, and consider its complement $Y = \mathbb{P}^d \setminus \cup\mathcal{A}$. Let $v_i \in K^{d+1}$ be a normal vector of H_i, so we have $H_i = \{x \in \mathbb{P}^d : v_i \cdot x = 0\}$. We assume that $\mathcal{A}$ is **essential**, i.e. the vectors $v_0, \dots, v_n$ span K^{d+1}, in other words the hyperplanes have no a common intersection point. Consider the algebraic torus $T^n = (K^*)^{n+1}/K^* \subseteq \mathbb{P}^n$, and define the map

$$\xi : Y \to T^n, \quad \xi(y) = (y \cdot v_0 : \cdots : y \cdot v_n).$$

We have that ξ maps Y to a linear subspace L of T^n. In fact, let $A \in K^{d+1,n+1}$ be the matrix whose columns are the vectors v_i. Let $B = (b_{ij}) \in K^{n-d,n+1}$ be the matrix whose rows are a basis for the kernel of (the linear map associated to) A. Let I_L be the ideal in $K[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$ generated by the linear forms $f = \sum_{j=0}^{n} b_{ij} x_j$ for $i \in \{1, \dots, n-d\}$.

Proposition 1.3.5 ([85, Proposition 4.1.1])**.** *The map ξ is an isomorphism between the arrangement complement Y and the linear subspace $L = V(I_L) \subseteq T^n$.*

The previous proposition gives us a correspondence between d-dimensional linear subspaces of T^n and essential hyperplane arrangements of $n+1$ hyperplanes in $\mathbb{P}^d$.

The **intersection lattice** of $\mathcal{A}$ is the poset

$$\mathcal{L}(\mathcal{A}) = \left\{ \bigcap_{i \in A} H_i : A \subseteq [n] \right\}$$

ordered by reversed inclusion. It is a lattice, and its elements are linear subspaces of $\mathbb{P}^d$. The **underlying matroid** of a hyperplane arrangement $\mathcal{A} = \{H_i : 0 \le i \le n\}$ is the simple matroid M of rank $d+1$ on the ground set $[n] = \{0, 1, \dots, n\}$ defined in the following way: a subset $A \subseteq [n]$ is independent if the corresponding normal vectors $\{v_i : i \in A\}$ are linearly independent. The lattice of flats $\mathcal{L}(M)$ of M is isomorphic to the intersection lattice of $\mathcal{A}$ [107, Proposition 3.6].

1.3.3 Wonderful compactifications

Let $\mathcal{A}$ be an essential hyperplane arrangement of $n+1$ hyperplanes in $\mathbb{P}^d$ and let $\mathcal{L}(A)$ be its intersection lattice. Fix a building set $\mathcal{G}$ of $\mathcal{L}(A)$. Let $\mathcal{G}^{op}$ denote the partially ordered set given by the set $\mathcal{G}$ with reverse inclusion. We provide here an alternative definition of De Concini-Procesi wonderful compactifications [31].

Definition 1.3.6 ([41, Definition 2.3])**.** Let $X_1, \ldots, X_t$ be a linear extension of $\mathcal{G}^{op}$. The De Concini-Procesi **wonderful compactification** $\overline{Y_{\mathcal{G}}}$ with respect to the building set $\mathcal{G}$ is the result of successively blowing up $\mathbb{P}^d$ at (the strict transforms of) $X_1, \ldots, X_t$.

Let M be the underlying matroid of $\mathcal{A}$. Then $\mathcal{G}$ can be viewed as a building set of the lattice of flats $\mathcal{L}(M)$ of M, since $\mathcal{L}(M) \simeq \mathcal{L}(A)$. Let $\Sigma = B_{\mathcal{G}}(M) \subseteq \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1} \simeq \mathbb{R}^n$ be the Bergman fan of M with respect to $\mathcal{G}$. Consider the n-dimensional toric variety X_Σ. From Proposition 1.3.5, the hyperplane arrangement complement $Y = \mathbb{P}^d \setminus \cup\mathcal{A}$ is isomorphic to a linear subspace of an n-dimensional algebraic torus T^n, which is naturally embedded in X_Σ. In summary we can embed Y inside the toric variety X_Σ and consider its closure $\overline{Y}$. Since by construction $\operatorname{trop}(Y) = |\Sigma|$ (see [85, Chapter 4]), $\overline{Y}$ is a tropical compactification of Y. This compactification coincides with the De Concini-Procesi wonderful compactification $\overline{Y}_{\mathcal{G}}$ (see [85, Section 6.7]).

In [42] Feichtner and Yuzvinsky showed that the cohomology of $\overline{Y}$ agrees with that of X_Σ. Since both varieties are *Homology Isomorphism schemes* (in the sense of the Definition in the Appendix of [76]), their Chow rings coincide as well.

Theorem 1.3.7 ([85, Theorem 6.7.14])**.** *Let $\overline{Y}_{\mathcal{G}}$ be a wonderful compactification of a hyperplane arrangement $\mathcal{A}$ with respect to a building set $\mathcal{G}$, and consider the toric variety X_Σ, where $\Sigma = B_{\mathcal{G}}(M)$ and M is the underlying matroid of $\mathcal{A}$. Then*

$$A^*(\overline{Y}_{\mathcal{G}}) \simeq A^*(X_\Sigma).$$

1.3.4 Effectivity of divisors of wonderful compactifications

In this section, we discuss an application of Algorithm 1.2.18 to wonderful compactifications. Let $\mathcal{A}$ be an essential hyperplane arrangement of $n+1$ hyperplanes in $\mathbb{P}^d$, fix a building set $\mathcal{G}$ of the intersection lattice $\mathcal{L}(A)$, and consider the wonderful compactification $\overline{Y}_{\mathcal{G}}$ of the complement $Y = \mathbb{P}^d \setminus \cup\mathcal{A}$ with respect to $\mathcal{G}$. From Proposition 1.3.5, there is an isomorphism ξ from Y to a linear subvariety L of T^n. Let $K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ be the coordinate ring of T^n, and let I_Y be the ideal in $K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ of L.

Proposition 1.3.8. *If $\overline{W}$ is an irreducible subvariety of $\overline{Y}_{\mathcal{G}}$ of codimension one such that $W = \overline{W} \cap Y \neq \emptyset$, then there exists $f \in K[x_1^{\pm1}, \ldots, x_n^{\pm1}]$ such that the ideal of $\xi(W)$ is $I_Y + (f)$.*

Proof. Recall that Y is a (very) affine variety. Let $\Gamma(W)$ and $\Gamma(Y)$ be the coordinate rings of W and Y respectively. Now by taking the coordinate rings from the diagram $W \hookrightarrow Y \xrightarrow{\xi} T^n$ we obtain the maps $K[x_1^{\pm1}, \ldots, x_n^{\pm1}] \to \Gamma(Y) \to \Gamma(W)$, where the first map has kernel I_Y. The kernel of the second map is a principal ideal, as $\Gamma(Y)$ is isomorphic to a Laurent ring, since Y is linear, and W is of codimension one inside Y. Now, since the ideal I_Y is linear, we obtain $\Gamma(W) \simeq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/(I_Y + (f))$ for some $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. □

Now let $\Sigma = B_{\mathcal{G}}(M)$ be the Bergman fan with respect to $\mathcal{G}$ of the underlying matroid M of $\mathcal{A}$. As we saw in the previous section, we can embed Y inside X_Σ. The closure of Y inside X_Σ coincides with $\overline{Y}_{\mathcal{G}}$. Now if $\overline{W}$ is an irreducible subvariety of $\overline{Y_{\mathcal{G}}}$ of codimension one with $W = \overline{W} \cap Y \neq \emptyset$, by the previous proposition, the ideal of W inside T^n is given by $I_Y + (f)$ for some polynomial $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Now we can use Algorithm 1.2.18 to compute the Chow class of $\overline{W}$ (which is the closure of W) inside X_Σ. Then, from Theorem 1.3.7 this class can be identified with the actual Chow class of $\overline{W}$ in $\overline{Y}_{\mathcal{G}}$. Note that Proposition 1.3.8 also implies that every (irreducible) effective divisor of $\overline{Y}_{\mathcal{G}}$ not contained in the boundary $\overline{Y}_{\mathcal{G}} \setminus Y$ is of this form. Thus, this whole procedure gives us a way to study the effective cone $\mathrm{Eff}^1(\overline{Y}_{\mathcal{G}})$ of effective divisors of $\overline{Y_{\mathcal{G}}}$.

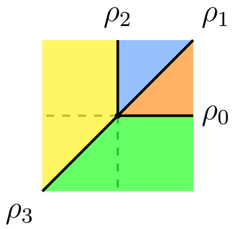
Remark 1.3.9. The effective cone of k-cycles of a variety X is usually defined inside $N^k(X)_{\mathbb{R}}$, whereas above we were talking about Chow classes (so inside $A^k(X)$). However, for wonderful compactifications, rational equivalence is the same as numerical equivalence, so the groups $N^k(X)$ and $A^k(X)$ are isomorphic, and the effective cone can be thought to be inside $A^k(X)_{\mathbb{R}}$. In fact, from a theorem of Keel (Theorem 2 in the appendix of [76]), since a wonderful compactification is the result of successive blow-ups of $\mathbb{P}^d$ at linear subspaces, rational equivalence is the same as homological equivalence. Then, by applying a theorem of Lieberman [81, Theorem 1], together with Theorem 1.3.7 and Poincaré duality, we have that numerical equivalence coincides with homological equivalence.

The reason why we restrict to effective divisors not contained in the boundary $\overline{Y}_{\mathcal{G}} \setminus Y$ is because the boundary is **divisorial**, that is, every irreducible component has codimension one. These components are called **boundary divisors**.

1.4 The Macaulay2 package TropicalToric.m2

The computations that we described in Section 1.2.4 were implemented in the package `TropicalToric.m2` [14] of the computer algebra system Macaulay2 [58]. In particular, this package implements toric cycles and the intersection product on simplicial toric varieties described in Section 1.2.1.

Example 1.4.1. Let X_Σ be the blow-up of $\mathbb{P}^2$ at one of the coordinate points, where the fan Σ and the first lattice points of its rays are as in Figure 1.1.



$$\begin{array}{cccc} \rho_0 & \rho_1 & \rho_2 & \rho_3 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Figure 1.1

Let H be the strict transform in X_Σ of a general line in $\mathbb{P}^2$ and E be the exceptional divisor. The Picard group of X_Σ is generated by the classes of these two divisors $\mathrm{Pic}(X_\Sigma) = \langle [H], [E] \rangle$. With the notation above, we have

$$[V(\rho_0)] = [H] - [E] \qquad [V(\rho_1)] = [E]$$
$$[V(\rho_2)] = [H] - [E] \qquad [V(\rho_3)] = [H]$$

Now we verify with our package that the divisor class $[V(\rho_1)]$ has negative self-intersection.

```
i1 : needsPackage "TropicalToric";
i2 : raysList = {{1,0},{1,1},{0,1},{-1,-1}};
i3 : coneList = {{0,1},{1,2},{2,3},{3,0}};
i4 : X = normalToricVariety (raysList, coneList);
```

Now define the toric cycle $V(\rho_1)$.

```
i5 : E = X_{1}
o5 = X
      {1}
o5 : ToricCycle on X
```

The type `ToricCycle` should not be confused with the type `ToricDivisor` from the `NormalToricVarieties` package. The toric cycle $V(\sigma)$ of the normal toric variety `X` associated to the cone σ given by a list of rays `L` is defined with the command `X_L`. For example, `X_{1,2}` or `X_{0}` define toric cycles, whereas `X_1` defines a toric divisor. We are allowed only to multiply a toric cycle with a toric divisor. Now, we finally compute the self intersection of E:

```
i6 : X_1 * E
o6 = - X
        {1, 2}
o6 : ToricCycle on X
```

The resulting cycle $-V(\rho_1 + \rho_2)$ is rationally equivalent to E^2. The negative sign tells us that the self-intersection number of the exceptional divisor is -1. We can compute the degree of maximal codimension cycles with `degCycle`:

```
i7 : degCycle(-X_{1,2})
o7 = -1
```

$\diamondsuit$

Furthermore, the function `classFromTropical` performs Algorithm 1.2.18 to compute a toric cycle rationally equivalent to a given irreducible subvariety Y of a simplicial toric variety X_Σ. The input of the function consists of the toric variety X_Σ and the ideal I of $Y \cap T^n$ of the Laurent ring of T^n. Since Laurent rings are not implemented in Macaulay2, the actual input will be instead the saturation of I with respect to the product of the variables in a polynomial ring:

```
i2 : X = toricProjectiveSpace 2;
i3 : R = QQ[x,y];
i4 : I = ideal(x+y+1);
i5 : classFromTropical(X,I)
o5 = X
      {0}
o5 : ToricCycle on X
i6 : J = ideal(x*y + x + y);
i7 : classFromTropical(X,J)
o7 = 2*X
        {0}
o7 : ToricCycle on X
```

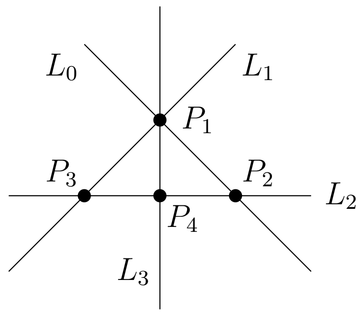
The function `classFromTropicalCox` allows us to input the ideal of Y in the Cox ring of X_Σ:

```
i8 : R = ring X;
i9 : I = ideal(R_0+R_1+R_2);
i10 : classFromTropicalCox(X,I)
o10 = X
       {0}
o10 : ToricCycle on X
```

The application of Algorithm 1.2.18 to wonderful compactifications, described in Section 1.3.4, is implemented in the function `classWonderfulCompactification`.

Example 1.4.2. Let $\mathcal{A}$ be a line arrangement consisting of 4 lines L_0, L_1, L_2, L_3 in $\mathbb{P}^2$ given by the equations $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, $x_0 + x_1 = 0$ respectively. Let A be the matrix with columns the normal vectors of the lines L_i, and let P_1, P_2, P_3, P_4 be the points of intersection of the lines of $\mathcal{A}$ as in the figure below.

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



The underlying matroid M of $\mathcal{A}$, on the ground set $\{0, 1, 2, 3\}$, is realized by the matrix A by labeling the columns with $0, 1, 2, 3$ respectively. The lattice of flats $\mathcal{L}(M)$ of M is represented by the diagram in Figure 1.2.

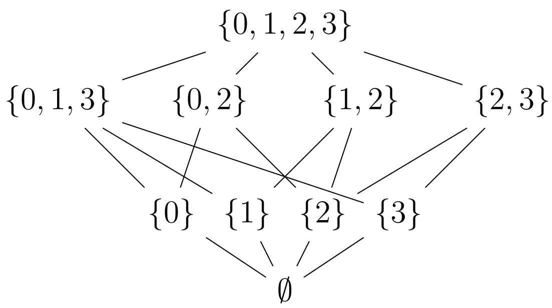


Figure 1.2

There are 4 rank 1 flats, corresponding to the lines L_0, L_1, L_2, L_3, and 4 rank 2 flats, corresponding to the points P_1, P_2, P_3, P_4. Let $\mathcal{G} = \mathcal{L}(M) \setminus \{\emptyset\}$ be the maximal building set of $\mathcal{L}(M)$. Then, the wonderful compactification $\overline{Y}$ of the complement $Y = \mathbb{P}^2 \setminus \cup\mathcal{A}$ with respect to $\mathcal{G}$ is the blow-up of $\mathbb{P}^2$ at the points P_1, P_2, P_3, P_4. In particular $\overline{Y}$ is a smooth projective surface, all Weil divisors are Cartier [60, Proposition II.6.11], and the class group is isomorphic to the Picard group [60, Corollary II.6.16]. From [60, Proposition V.3.2], the Picard group of $\overline{Y}$ has a basis given by

$$\operatorname{Pic}(\overline{Y}) = \langle [H], [E_1], \ldots, [E_4] \rangle, \tag{1.7}$$

where $[H]$ is the class of the strict transform H of a general line in $\mathbb{P}^2$, and $[E_i]$ is the class of the exceptional divisor E_i of the blow-up at P_i.

The Bergman fan $\Sigma \subseteq \mathbb{R}^4/\mathbb{R}\mathbf{1}$ of M with respect to $\mathcal{G}$ has 8 rays. We denote them by $\{\rho_i : 0 \leq i \leq 7\}$. Their first lattice points are given by the columns of the following matrix

$$\begin{array}{cccccccc} \rho_0 & \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \rho_6 & \rho_7 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

where $\rho_0, \rho_1, \rho_2, \rho_3$ correspond to the rank 1 flats in $\mathcal{G}$, which in turn correspond to the lines L_0, L_1, L_2, L_3 respectively, and $\rho_4, \rho_5, \rho_6, \rho_7$ correspond to the rank 2 flats in $\mathcal{G}$, that correspond to the points P_1, P_2, P_3, P_4 respectively. Since $\mathcal{G}$ is the maximal building set, the maximal cones of Σ are just the maximal chains of the lattice of flats $\mathcal{L}(M)$.

By using the isomorphism in Theorem 1.3.7, let $[Y_{\rho_i}]$ denote the class in $A^*(\overline{Y})$ isomorphic to the class of the torus invariant divisor of X_Σ associated to the ray ρ_i. Expressing these divisors in the Picard basis (1.7) we have:

$$\begin{aligned} &[Y_{\rho_0}] = [H] - [E_1] - [E_2] && [Y_{\rho_4}] = [E_1] \\ &[Y_{\rho_1}] = [H] - [E_1] - [E_3] && [Y_{\rho_5}] = [E_2] \\ &[Y_{\rho_2}] = [H] - [E_2] - [E_3] - [E_4] && [Y_{\rho_6}] = [E_3] \\ &[Y_{\rho_3}] = [H] - [E_1] - [E_4] && [Y_{\rho_7}] = [E_4] \end{aligned} \tag{1.8}$$

Let K be some algebraically closed field of characteristic zero, and let $K[y_0^{\pm 1}, y_1^{\pm 1}, y_2^{\pm 1}]$ be the Laurent ring of the torus

$$T^3 = \{(1 : y_0 : y_1 : y_2) : y_0, y_1, y_2 \in K^*\} \subseteq \mathbb{P}^3.$$

The embedding $Y \hookrightarrow T^3$ is given by $(x_0 : x_1 : x_2) \mapsto (x_0 : x_1 : x_2 : x_0 + x_1)$, and the Laurent ideal of Y inside T^3 is $I = (-1 - y_0 + y_2)$.

Now let C be the conic in $\mathbb{P}^2$ passing through P_1, P_2 and P_3 given by the equation $x_0x_1 + x_0x_2 + x_1x_2$. The ideal of C in T^3 is $(y_0 + y_1 + y_0y_1) + I$. We expect the class of its strict transform in $\overline{Y}$ to be $[2H - E_1 - E_2 - E_3]$. We now verify this with our package, using the function `classWonderfulCompactification`:

```
i2 : R = QQ[y_0,y_1,y_2];
i3 : I = ideal(-1-y_0+y_2);
i4 : f = y_0+y_1+y_0*y_1;
i5 : raysList = {{-1,-1,-1},{1,0,0},{0,1,0},
                 {0,0,1},{0,-1,0},{-1,0,-1},
                 {1,1,0},{0,1,1}};
```

```
i6 : conesList = {{4,0},{4,1},{4,3},{5,0},{5,2},
                  {6,1},{6,2},{7,2},{7,3}};
i7 : X = normalToricVariety (raysList, conesList);
i8 : D = classWonderfulCompactification(X,I,f)
o8 = X    + X    + X
      {0}    {4}    {1}
o8 : ToricCycle on X
```

To check that this is the result we expect, compare with (1.8). Note that we have (tropically) dehomogenized the rays of X_Σ with respect to the first coordinate in order to be consistent with our choice of coordinates of T^3. $\diamondsuit$

1.5 An application to the moduli space $\overline{M}_{0,n}$

1.5.1 Fulton conjecture

Let M_g be the $(3g-3)$-dimensional variety that parametrizes smooth curves of genus g. Of course M_g is not compact, since smooth curves degenerate to singular ones. However, there exists a canonical compactification, the so-called *Deligne-Mumford compactification* $\overline{M}_g$ that parametrizes *stable* curves: curves with at most ordinary nodes as singularities and finite automorphism groups.

In order to better describe the boundary $\partial\overline{M}_g = \overline{M}_g \setminus M_g$, it is useful to introduce moduli spaces of pointed curves $\overline{M}_{g,n}$ parametrizing $(C; p_1, \ldots, p_n)$, where $p_1, \ldots, p_n$ are distinct smooth points on the nodal curve C and there are only finitely many automorphisms of C fixing $p_1, \ldots, p_n$. From this definition it follows that $\overline{M}_g = \overline{M}_{g,0}$.

The boundary of $\overline{M}_{g,n}$ is a codimension one subvariety Δ with components

$$\Delta = \bigcup_{\substack{0\le g'<g \\ 0\le n'<n}} \Delta_{g',n'},$$

where $\Delta_{g',n'}$ is the image of a natural gluing map from $\overline{M}_{g-1,n+2}$ for $g' = 0$, and from $\overline{M}_{g'+1,n'} \times \overline{M}_{g-g'+1,n-n'}$ for $g' > 0$.

Definition 1.5.1. We say that a scheme X is **stratified** by a finite collection of irreducible, locally closed subschemes U_i if X is a disjoint union of the U_i and, in addition, the closure of any U_i is a union of U_j. The sets $\overline{U_i}$ are called the (closed) **strata** of the stratification.

The variety $\overline{M}_{g,n}$ has a stratification, where the codimension k strata are the irreducible components of the locus parametrizing pointed curves with at least k singular points. A nef k-cycle must intersect any codimension k strata nonnegatively. It is thus natural to consider the following conjecture:

Conjecture 1.5.2 ($F_k(g,n)$)**.** *The following equivalent statements hold true.*

1. *A codimension-k cycle on $\overline{M}_{g,n}$ is nef if and only if it nonnegatively intersects any dimension k strata.*
2. *The cone $\mathrm{Eff}_k(\overline{M}_{g,n})$ is generated by the k-dimensional strata.*

Conjecture 1.5.2 will be denoted by $F_k(g,n) = F^{n+3g-3-k}(g,n)$ (note that $\dim \overline{M}_{g,n} = n+3g-3$), and it is sometimes referred as *Fulton conjecture* (or *Fulton question*). One of the intuitions behind this conjecture was that $\overline{M}_{0,n}$ is "similar to a toric variety" and for toric varieties the effective cone of divisors is indeed generated by the (classes of) torus invariant divisors, which are the analogue of the codimension one strata.

In general, $F_k(g,n)$ is false. For some examples for $g \geq 1$, see for instance [89]. However, for several cases the conjecture is still open. The following theorem makes the case $g = 0$ particularly interesting.

Theorem 1.5.3 (Bridge theorem [55])**.**

$$F_1(g,n)\ \forall g,n \geq 0 \Longleftrightarrow F_1(0,n)\ \forall n \geq 3.$$

Now we list a series of facts about $F_k(0,n)$:

- $F_1(0,n)$ is true for $n \leq 7$ [77].
- $F_1(0,n)$ for $n > 7$ is still open.
- $F^1(0,n)$ is true for $n \leq 5$.
- $F^1(0,n)$ is false for $n \geq 6$ [110]. However, $\mathrm{Eff}^1(\overline{M}_{0,6})$ is generated by the codimension 1 strata and the 15 counterexamples found in [110] (called *Keel-Vermeire divisors*), see [61].
- $F_k(0,n)$ is false for $1 < k < n-3$ for $n \gg 0$ (this follows from the previous statement by lifting the Keel-Vermeire divisors, see [101]).

We now focus on $F^1(0,n)$. As we saw above, Fulton conjecture in this case is false for $n \geq 6$. However, for $n = 6$ the effective cone $\mathrm{Eff}^1(\overline{M}_{0,n})$ is still polyhedral. Therefore it is natural to ask for which n the effective cone $\mathrm{Eff}^1(\overline{M}_{0,n})$ of divisors of $\overline{M}_{0,n}$ is polyhedral. It was proved in [21] that for $n \geq 10$ the effective cone is not polyhedral. To the author's knowledge, the cases $7 \leq n \leq 9$ are still open.

Finally, we note that in [22] the effective cone $\mathrm{Eff}^1(\overline{M}_{0,n})$ was conjectured to be generated by the boundary divisors (the codimension one strata), and a class of divisors called *hypertree divisors*. Some counterexamples to this conjecture, even in the cases $7 \leq n \leq 9$, were independently found in [91] and [33].

1.5.2 Computations on effective divisors of $\overline{M}_{0,7}$

In this section, we discuss some computations that were performed to find effective divisors of $\overline{M}_{0,7}$ with the package `TropicalToric.m2`, with a view towards the problem of determining the effective cone of $\overline{M}_{0,n}$, in particular for the case $n = 7$. The computations were run on the machine Galois (galois.warwick.ac.uk) in a brute-force search form.

The Deligne-Mumford compactification of $M_{0,n}$ can be viewed as a wonderful compactification. The construction of $\overline{M}_{0,n}$ as successive blow-ups is called Kapranov construction [72]. The underlying matroid associated to $\overline{M}_{0,n}$ (in the sense discussed in Section 1.3.3) is the graphic matroid of the complete graph K_{n-1}.

Consider the complete graph K_6, and let A be the following realization matrix for its matroid

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Each row of A is associated to a vertex of K_6, that we number from 1 to 6, and each column to be associated with one edge, for instance the first column is associated to the edge $\{1, 6\}$ of K_6.

Let $S = \mathbb{C}[z_0, \ldots, z_{14}]$ be the coordinate ring of $\mathbb{P}^{14}$. Each variable z_i is associated to the edge of K_6 corresponding to the i-th column of the matrix A above. For instance z_0 is associated to the edge $\{1, 6\}$ of K_6. Now let $R = \mathbb{C}[x_0^{\pm 1}, \ldots, x_{13}^{\pm 1}]$ be the Laurent ring of the torus T^{14} inside $\mathbb{P}^{14}$, where here we are setting $x_i = z_i/z_{14}$.

Let I be the ideal of R generated by linear polynomials whose coefficients are given by the rows of the kernel of (the linear map associated to) the matrix A:

$$\begin{aligned} I =& \langle -x_0 + x_4 + x_5,\ -x_1 + x_4 + x_6,\ -x_2 + x_4 + x_7,\ -x_3 + x_4 + x_8,\ -x_0 + x_3 + x_9, \\ & - x_1 + x_3 + x_{10},\ -x_2 + x_3 + x_{11},\ -x_0 + x_2 + x_{12},\ -x_1 + x_2 + x_{13},\ -x_0 + x_1 + 1 \rangle \subseteq R \end{aligned}$$

The ideal I is the ideal of $M_{0,7} \subseteq T^{14}$. Let $M(K_6)$ be the graphic matroid of K_6. By construction, $M(K_6)$ is realized by the matrix A, and it is the underlying matroid of $M_{0,7}$. Let X_Σ be the toric variety with fan Σ given by the Bergman fan of $M(K_6)$ with the fan structure given by the minimal building set.

From Proposition 1.3.8, each prime divisor D of $\overline{M}_{0,7}$ corresponds to a polynomial $f \in R$ and its class $[D] \in A^1(\overline{M}_{0,7})$ can be computed from the tropicalization of the very affine variety with ideal $I + (f) \subseteq R$.

Remark 1.5.4. Each polynomial $f \in R$ has a representative modulo I expressed just in terms of the first 5 variables. Further, we can choose this representative homogeneous, since $1 \equiv x_0 - x_1$ modulo I.

Remark 1.5.5. Fix $f \in R$ homogeneous of degree d and set $J = I + (f)$. The tropicalization $\mathrm{trop}(V(J))$ depends just on the matroid of the vector space J_d, since J has a tropical basis of degree at most d (see [2, Theorem 3.7]). From this fact, it follows that as f varies among homogeneous polynomials of degree d of R, there are finitely many possibilities for $\mathrm{trop}(J)$ (and therefore for the class $[D]$ of the associated prime divisor D). Further, we can choose the coefficients of f to be rationals, and thus integers, as, once the matroid is fixed, the possibilities for f range in a hyperplane arrangement complement with rational coefficients.

Brute-force search We performed a brute-force search of effective divisors of $\overline{M}_{0,7}$ by computing the classes of prime divisors D associated to homogeneous polynomials $f \in R$ of degree d. From Remark 1.5.5 we could assume that f had integer coefficients.

S_n-action and orbits We recall that $M_{0,n}$ has a natural action of S_n that simply permutes the n marked points. This action can be naturally extended to $\overline{M}_{0,n}$. The action on the boundary divisors δ_I, indexed by subsets $I \subseteq \{1, \dots, n\}$ such that $|I|, |I^c| \geq 2$, acts by permuting the elements of I, so that if $\sigma \in S_n$, then $\sigma \cdot \delta_I = \delta_{\sigma(I)}$.

$\mathbf{d = 2}$ For $d = 2$, we computed the classes associated to more than $60,000$ polynomials. We then checked, interfacing with Polymake [54], which of the classes obtained lie outside the cone generated by the boundary divisors. Then, we computed the orbits (with respect to the S_n-action described above) of these classes. We found 135 classes, divided in two orbits, 75 of them inside an orbit of cardinality 105, and the remaining 60 contained in another orbit of cardinality 420. These two orbits were recognized to be: the first, the orbit of the pullbacks of the Keel-Vermeire divisors (i.e. pullbacks of hypertree divisors for $n = 6$) and the second the orbit of the hypertree divisors (for $n = 7$).

Orbits found outside the boundary cone for $d = 2$	
pullbacks of Keel-Vermeire divisors	$105 = 7 * 15$
hypertree divisors (for n=7)	$420 = 7 * 60$

d = 3 We then run some computations for $d = 3$. The classes of more than $10,000$ polynomials were computed. A total of 20 classes outside the boundary cone were found, and they belong to the same orbit of cardinality 420. This orbit was recognized to be the orbit of the example found by Opie in [91].

Orbits found outside the boundary cone for $d = 3$	
Opie's example	$420 = 7 * 60$

It is very likely that by running the same computations for $d = 4$, one could find the (orbit of the) degree 4 example found in [33].

In addition, we note that new examples of extremal rays of the effective cone of $\overline{M}_{0,7}$ were found in [104] on the order of 100,000. This contrasts significantly with $\overline{M}_{0,6}$, in which the extremal rays of the effective cone are just 40.

Chapter 2

Weierstrass sets on finite graphs

2.1 Introduction

Let X be a smooth projective algebraic curve of genus g and fix a point $P \in X$. Denote by $H(P)$ the set of pole orders at P of rational functions regular on $X \setminus \{P\}$. By the Weierstrass gap theorem (see [40, III.5.3]), the set of *gaps* $G(P) = \mathbb{N} \setminus H(P)$ has cardinality exactly g. This implies that $H(P)$ is a *numerical semigroup*, that is, a cofinite additive submonoid of $\mathbb{N}$. The numerical semigroups arising in this way are called *Weierstrass semigroups*. We have $G(P) = \{1, \dots, g\}$ except in a finite number of points, called *Weierstrass points* of X (see [40, III.5.9]).

In 1893 Hurwitz [64] asks if all the numerical semigroups arise in this manner. Several years later, in 1980, Buchweitz [20] showed that the numerical semigroup

$$S = \langle 13, 14, 15, 16, 17, 18, 20, 22, 23 \rangle$$

is not Weierstrass (see also [36, page 499]). The proof essentially gives the following necessary condition for a semigroup to be Weierstrass: the m-sumset of the set of gaps must satisfy $|mG(P)| \leq (2m-1)(g-1)$ for any integer $m \geq 2$. Several numerical semigroups not satisfying the previous condition are constructed in [78]. Furthermore, in [37] it was proved that for a fixed numerical semigroup S, the set of integers m that do not satisfy the above condition is finite. Despite these results, little is known more generally about the family of Weierstrass semigroups. For instance, the problem of determining its density in the set of all numerical semigroups is still open [71].

After the advent of tropical geometry, the tropical analogues of many classical results in algebraic geometry were found. Baker and Norine [7] proved a Riemann-Roch theorem for graphs, which was successively extended by Gathmann and Kerber [50] and Mikhalkin and Zharkov [87] to *metric graphs*, namely (abstract) tropical curves. The analogue notion of Weierstrass points on graphs was studied for instance in [6, Section 4] and [99].

Inspired by a work of Kang, Matthews and Peachey [70], in this paper we investigate possible tropical analogues of Weierstrass semigroups. We will focus our attention on graphs rather than metric graphs, the latter are left for future work. It was already noted in [70] that two possible non-equivalent definitions can be given, as we now explain. Throughout this paper, a *graph* will mean a finite connected multigraph having no loop edges. Let G be a graph and fix a vertex $P \in V(G)$ of G. The *functional Weierstrass set* of G at P is defined by

$$H_f(P) = \{n \in \mathbb{N} : \exists f \in \mathcal{M}(G) \text{ that has a unique pole of order } n \text{ at } P\},$$

where $\mathcal{M}(G)$ is the set of all integer-valued functions on the vertices of G. The *rank Weierstrass set* of G at P is defined by

$$H_r(P) = \{n \in \mathbb{N} : r(nP) > r((n-1)P)\},$$

where $r(D)$ denotes the *rank* of the divisor D of the graph G, in the sense of Baker and Norine [7] (see Section 2.2). Classically, for curves, we have $H_f(P) = H_r(P) = H(P)$. However this is not the case for graphs, for instance the cardinality of the set difference $H_f(P) \setminus H_r(P)$ can be arbitrarily large [70, Proposition 3.9].

Our first main result was conjectured in [70], and relates the two sets when G is a graph with no multiple edges and more than one vertex, that in this paper will be called *simple*.

Theorem A (Theorem 2.3.4)**.** *Let G be a simple graph. For every $P \in V(G)$ we have $H_r(P) \subseteq H_f(P)$.*

As an application of the previous theorem, we calculate the rank and functional Weierstrass set of the graphs K_{n+1} and $K_{n,m}$.

Secondly, we completely characterize the subsets of $\mathbb{N}$ arising as functional Weierstrass sets of graphs and of simple graphs, answering a question in [70].

Theorem B (Theorem 2.4.5)**.** *The functional Weierstrass sets of graphs (resp. simple graphs) are precisely the additive submonoids of $\mathbb{N}$ (resp. numerical semigroups).*

Further, we give a sufficient condition for a subset of $\mathbb{N}$ to be the rank Weierstrass set of a graph.

Theorem C (Theorem 2.5.3). *Let $e_1 \geq e_2 \geq \cdots \geq e_n \geq 0$ be integers and set $s_i = \sum_{j=1}^{i} e_j$. There exists a simple graph G with a vertex $P \in V(G)$ such that*

$$H_r(P) = \{0, s_1, s_2, \ldots, s_{n-1}\} \cup (s_n + \mathbb{N}).$$

The previous theorem allows us to construct families of graphs in which the rank Weierstrass set is not a semigroup (see Example 2.5.6), justifying the name "Weierstrass *set*".

2.2 Preliminaries

In this section, we fix our notation and review the basics and some results of Riemann-Roch theory on finite graphs.

In this paper, a *graph* will mean a finite connected multigraph having no loop edges; a *simple graph* will mean a graph with no multiple edges and more than one vertex. Let G be a graph and let $V(G)$ (resp. $E(G)$) denote the set of vertices (resp. edges) of G. The set $\operatorname{Div}(G)$ of *divisors* of G is the free abelian group on $V(G)$. We think of a divisor as a formal integer linear combination of the vertices $D = \sum_{P \in V(G)} a_P P \in \operatorname{Div}(G)$ with $a_P \in \mathbb{Z}$. For convenience, we will write $D(P)$ for the coefficient a_P of P in D. The *degree* of a divisor D is defined by $\deg(D) = \sum_{P \in V(G)} D(P) \in \mathbb{Z}$. If $D, D' \in \operatorname{Div}(G)$ are two divisors, then $D \geq D'$ if and only if $D(P) \geq D'(P)$ for all $P \in V(G)$. A divisor D is *effective* if $D \geq 0$. The set of effective divisors of degree d is denoted by $\operatorname{Div}_+^d(G)$.

Let $\mathcal{M}(G) = \operatorname{Hom}(V(G), \mathbb{Z})$ be the set of integer-valued functions on the vertices of G. For every vertex $P \in V(G)$, define the *indicator function* $f_P \in \mathcal{M}(G)$ by

$$f_P(Q) = \begin{cases} -1 & Q = P, \\ 0 & Q \neq P. \end{cases}$$

Let $f \in \mathcal{M}(G)$, and denote by $\mathcal{N}(P)$ the neighbourhood of $P \in V(G)$, that is, the subset of vertices of G adjacent to P. Define the *Laplacian operator* $\Delta : \mathcal{M}(G) \to \operatorname{Div}(G)$ by

$$\Delta f = \sum_{P \in V(G)} \left(\sum_{Q \in \mathcal{N}(P)} \big(f(P) - f(Q)\big) \right) P.$$

The divisors of the form Δf are *principal.* For convenience we will write $\Delta_P f$ for the coefficient $\Delta f(P)$. If we think of f as a vector, the Laplacian operator can be seen as the multiplication of the *Laplacian matrix* $Q = D - A$, where D is the diagonal matrix of the degrees of the vertices, and A is the adjacency matrix of G. The matrix Q has rank

$|V(G)|-1$, and $\ker Q = (1,\ldots,1)^t$. From this fact, it is easy to see that every principal divisor has degree 0.

Two divisors $D, D' \in \mathrm{Div}(G)$ are *linearly equivalent*, written $D \sim D'$, if $D - D' = \Delta f$, for some $f \in \mathcal{M}(G)$. The *linear system* associated to a divisor $D \in \mathrm{Div}(G)$ is

$$|D| = \{E \in \mathrm{Div}(G) : E \sim D, E \geq 0\}.$$

The *rank* $r(D)$ of a divisor D is defined as -1 if $|D| = \emptyset$, otherwise

$$r(D) = \max\{k \in \mathbb{N} : |D - E| \neq \emptyset, \forall E \in \mathrm{Div}_+^k(G)\}.$$

Lemma 2.2.1. *[7, Lemma 2.1] For all $D_1, D_2 \in \mathrm{Div}(G)$ with $r(D_1), r(D_2) \geq 0$ we have $r(D_1 + D_2) \geq r(D_1) + r(D_2)$.*

Lemma 2.2.2. *[6, Lemma 2.7] Let G be a graph, and let $D \in \mathrm{Div}(G)$. Then $r(D-P) \geq r(D) - 1$ for all $P \in V(G)$, and if $r(D) \geq 0$, then $r(D - P) = r(D) - 1$ for some $P \in V(G)$.*

The *canonical divisor* of G is

$$K_G = \sum_{P \in V(G)} (\deg(P) - 2)P.$$

It has degree $\deg(K_G) = 2g-2$, where $g = |E(G)|-|V(G)|+1$ is the *genus* (or *cyclomatic number*) of the graph G. We are now ready to state the Riemann-Roch theorem for graphs, proved by Baker and Norine [7].

Theorem 2.2.3 (Riemann-Roch for graphs)**.** *Let D be a divisor on a graph G of genus g. Then*

$$r(D) - r(K_G - D) = \deg(D) + 1 - g.$$

For $A \subseteq V(G)$ and $Q \in A$, let $\mathrm{outdeg}_A(Q)$ denote the number of edges incident with Q and a vertex in $V(G) \setminus A$. Fix $P \in V(G)$. A divisor D is *P-reduced* if it is effective in $V(G) \setminus \{P\}$, and every non-empty subset $A \subseteq V(G) \setminus \{P\}$ contains a vertex $Q \in A$ such that $\mathrm{outdeg}_A(Q) > D(Q)$.

Proposition 2.2.4. *[7, Proposition 3.1] Let P be a vertex of a graph G. For every divisor D in G, there exists a unique P-reduced divisor D' such that $D \sim D'$.*

Following [6, Section 4], a vertex $P \in V(G)$ of a graph G of genus g, is a *Weierstrass point* if $r(gP) \geq 1$. We now state an analogue of the Weierstrass gap theorem for graphs.

Lemma 2.2.5. *[6, Lemma 4.2] Let G be a graph of genus g, and fix a vertex $P \in V(G)$.*

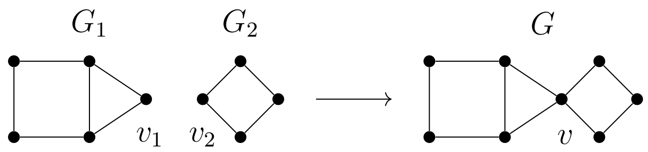
1. *P is a Weierstrass point if and only if* $\mathbb{N} \setminus H_r(P) \neq \{1, \ldots, g\}$.

2. $|\mathbb{N} \setminus H_r(P)| = g$.

3. $\mathbb{N} \setminus H_r(P) \subseteq \{1, 2, \ldots, 2g-1\}$.

Note that, in the classical case for curves, the inclusion $\mathbb{N} \setminus H(P) \subseteq \{1, \ldots, 2g-1\}$ follows from $|\mathbb{N} \setminus H(P)| = g$ and the fact that $H(P)$ is a semigroup (see [100, Lemma 2.14]).

We now describe a binary operation on graphs that we will use frequently in Section 2.4 and 2.5. Let G_1 and G_2 be two graphs and v_1 and v_2 be vertices of respectively G_1 and G_2. The *vertex gluing* (or *vertex identification*) of v_1 and v_2 is the graph G obtained from G_1 and G_2 by identifying v_1 and v_2 as a new vertex v.



2.3 The inclusion $H_r(P) \subseteq H_f(P)$

In this section, we will assume that G is a simple graph. We will prove the inclusion $H_r(P) \subseteq H_f(P)$ for every vertex $P \in V(G)$. First, we will need a series of lemmas, inspired by the Cori-Le Borgne algorithm [27, Proposition 2] for the rank of divisors of a complete graph.

Lemma 2.3.1. *Fix a vertex $P \in V(G)$ and let D be a P-reduced divisor on G. There exists a neighbour $Q \in V(G) \setminus \{P\}$ of P such that $D(Q) = 0$.*

Proof. Set $A = V(G) \setminus \{P\}$ and let $\mathcal{N}(P) \subseteq A$ be the set of neighbours of P. Assume by contradiction that $D(Q) \geq 1$ for all $Q \in \mathcal{N}(P)$. Since G is simple, we have $\text{outdeg}_A(Q) = 1$ for all $Q \in \mathcal{N}(P)$. This implies $D(Q) \geq \text{outdeg}_A(Q)$ for all $Q \in A$, contradicting the fact that D is P-reduced. $\square$

Let D be a divisor on G of rank r. Using the same terminology as in [27, 30], a *proof* for the rank of D is an effective divisor E of degree $r+1$ with $|D - E| = \emptyset$. We denote by $\text{Proof}(D)$ the set of proofs of D. Note that if $D \sim D'$, then $\text{Proof}(D) = \text{Proof}(D')$.

Lemma 2.3.2. *Fix a vertex $P \in V(G)$ and let D be a P-reduced divisor on G of rank zero. We have* $\text{Proof}(D) \setminus \{P\} \neq \emptyset$.

Proof. If $D(P) > 0$, then $P \notin \text{Proof}(D) \neq \emptyset$. Now assume $D(P) = 0$, from Lemma 2.3.1 there exists a neighbour Q of P such that $D(Q) = 0$. The divisor $D' = D - Q$ is Q-reduced. In fact, let $A \subseteq V(G) \setminus \{Q\}$: if $P \notin A$ then, since D is P-reduced, we have $\text{outdeg}_A(v) > D(v) = D'(v)$ for some $v \in A$; otherwise if $P \in A$, then $\text{outdeg}_A(P) \geq 1 > 0 = D(P) = D'(P)$. Finally, since $D'(Q) < 0$ and D' is Q-reduced, it follows that Q is a proof for D with $Q \neq P$. □

Lemma 2.3.3. *Let D be a divisor on G. For every vertex $P \in V(G)$ there exists $E \in$ $\text{Proof}(D)$ such that $E(P) = 0$.*

Proof. Without loss of generality, we can assume that D is P-reduced. We proceed by induction on the rank r of D. The case $r = -1$ is trivial. If $r = 0$ the assertion follows from Lemma 2.3.2.

Now suppose that D has rank $r \geq 1$ and assume the statement for divisors of rank $r - 1$. From Lemma 2.2.2 we have $r(D - P') = r - 1$ for some vertex $P' \in V(G)$. By the inductive hypothesis there exists $E' \in \text{Proof}(D - P')$ such that $E'(P) = 0$. Now apply Lemma 2.3.2 to the P-reduced divisor equivalent to $D - E'$. Thus there exists $Q \in \text{Proof}(D - E')$ with $Q \neq P$. We conclude by noting that $E = E' + Q \in \text{Proof}(D)$ and $E(P) = 0$. □

Now we prove the main result of the section. We will follow the proof outlined in [70, Theorem 2.4] in which the previous Lemma 2.3.3 was the key step missing.

Theorem 2.3.4. *Let G be a simple graph. For every $P \in V(G)$ we have $H_r(P) \subseteq H_f(P)$.*

Proof. Let $n \in H_r(P)$. By Lemma 2.3.3, there exists an effective divisor $E \in \text{Proof}((n-1)P)$ such that $E(P) = 0$. By the choice of E and since $r(nP) > r((n-1)P)$, there exists a function $f \in \mathcal{M}(G)$ such that

$$\begin{aligned} (n-1)P - E + \Delta f &\ngeq 0, \\ nP - E + \Delta f &\geq 0. \end{aligned}$$

This, together with the fact that $E(P) = 0$, implies that f has a unique pole of order n at P, that is $n \in H_f(P)$. □

Remark 2.3.5. In general, Theorem 2.3.4 fails when G has just one vertex P (in which case we have $H_f(P) = \{0\}$ and $H_r(P) = \mathbb{N}$) and when G has multiple edges. An example of the last statement is given by the multigraph B_n with two vertices connected by n edges. For every vertex $P \in V(B_n)$, it results $H_f(P) = n\mathbb{N}$ and $H_r(P) = \mathbb{N} \setminus \{1, \ldots, n-1\}$, hence $H_r(P) \nsubseteq H_f(P)$.

Following the strategy outlined in [70], as an application of Theorem 2.3.4 we calculate the rank Weierstrass set of complete and complete bipartite graphs from their functional Weierstrass set. In fact, in these two cases we have $H_r(P) = H_f(P)$ for every vertex P of the graph.

Lemma 2.3.6. *[70, Porism 2.11] Let G be a simple graph, let $P \in V(G)$ be a vertex and let $G - P$ be the graph G with the vertex P and its adjacent edges removed. If $G - P$ is connected and $f \in \mathcal{M}(G)$ is a function with a unique pole at P, then $f(P) < f(Q)$ for every $Q \in V(G)$.*

Let $n \geq 1$ and consider the complete graph K_{n+1}.

Lemma 2.3.7. *[70, Proposition 3.7] For every vertex $P \in V(K_{n+1})$, we have $H_f(P) = \langle n, n+1 \rangle$.*

Now we deal with the analogous result for the rank.

Corollary 2.3.8. *For every vertex $P \in V(K_{n+1})$, we have $H_r(P) = \langle n, n+1 \rangle$.*

Proof. By Lemma 2.3.7 and Theorem 2.3.4 we have $H_r(P) \subseteq H_f(P) = \langle n, n+1 \rangle$. Finally, from Lemma 2.2.5 we have $|\mathbb{N} \setminus H_r(P)| = g(K_{n+1}) = |\mathbb{N} \setminus \langle n, n+1 \rangle|$. □

Now let $n, m \geq 1$ and consider the complete bipartite graph $K_{n.m}$. The proof of the following lemma is inspired by the proof of [70, Proposition 3.7].

Lemma 2.3.9. *Let $P \in V(K_{m,n})$ be a vertex of degree n, we have*

$$H_f(P) = n\mathbb{N} \cup (n(m-1) + \mathbb{N})$$

Proof. If n or m is equal to 1, then $H_f(P) = \mathbb{N}$, so we assume that $n, m \geq 2$. We label the vertices of $K_{n,m}$ of degree n by $P = P_1, P_2, \ldots, P_m$ and the vertices of degree m by $Q = Q_1, \ldots, Q_n$. Let $f \in \mathcal{M}(K_{n,m})$ with a unique pole at P. By Lemma 2.3.6 the minimum of f is attained at P. Without loss of generality we can assume $f(P) = 0$. Set $f(Q_i) = a + \alpha_i$ for $i \in \{1, \ldots, n\}$ with $a, \alpha_i \in \mathbb{N}$ and $\alpha_1 = 0$, and $f(P_i) = b + \beta_i$ for $i \in \{2, \ldots, m\}$ with $b, \beta_i \in \mathbb{N}$ and $\beta_2 = 0$. Now we have

$$\begin{aligned}
-\Delta_P f &= na + \sum_{i=2}^{n} \alpha_i \geq 0, \\
\Delta_Q f &= a + (m-1)(a-b) - \sum_{i=3}^{m} \beta_i \geq 0, \\
\Delta_{P_2} f &= n(b-a) - \sum_{i=2}^{n} \alpha_i \geq 0.
\end{aligned}$$

Now if $a \geq b$, then from the third inequality $0 \geq n(b-a) \geq \sum \alpha_i \geq 0$. Hence $\alpha_i = 0$ for all $i \in \{2, \ldots, m\}$ and $-\Delta_P f = na \in n\mathbb{N}$. On the other hand, if $a < b$, then from the second inequality

$$a + (m-1)(a-b) \geq \sum \beta_i \geq 0 \Rightarrow a \geq (m-1)(b-a) \geq m-1.$$

This implies $-\Delta_P f = na + \sum \alpha_i \geq n(m-1)$, that is $-\Delta_P f \in n(m-1) + \mathbb{N}$. This proves the inclusion $H_f(P) \subseteq n\mathbb{N} \cup (n(m-1) + \mathbb{N})$.

For the reverse inclusion, it is enough to note that, for the indicator function f_P, we have $\Delta f_P = -nP + \sum Q_i$. In addition, for every $t \in \{1, \ldots, n-1\}$

$$\Delta \left(m f_P + \sum_{i=1}^{t} f_{Q_i} \right) = -\big(n(m-1) + t\big)P + \sum_{i=2}^{m} tP_i + \sum_{i=t+1}^{n} mQ_i.$$

□

Proceeding similarly as in the proof of Corollary 2.3.8, we are able to calculate the rank Weierstrass set of complete bipartite graphs.

Corollary 2.3.10. *Let $P \in V(K_{m,n})$ be a vertex of degree n, we have*

$$H_r(P) = n\mathbb{N} \cup ((m-1)n + \mathbb{N})$$

Remark 2.3.11. The computation of the rank Weierstrass set of complete graphs (Corollary 2.3.8) was already implicit in the proof of [26, Theorem 8], a result that gives an upper bound for the gonality sequence of complete graphs. In fact, we note that the rank Weierstrass set of a complete graph coincides with its gonality sequence.

Question 2.3.12. Under which conditions on the graph G and the vertex P do we have $H_f(P) = H_r(P)$?

2.4 Functional Weierstrass sets

In this section, we characterize the subsets of $\mathbb{N}$ that arise as the functional Weierstrass sets of some graph or simple graph.

Lemma 2.4.1. *Let G be a graph and fix a vertex $P \in V(G)$. The functional Weierstrass set $H_f(P)$ is an additive submonoid of $\mathbb{N}$. Further, if G is simple, then $H_f(P)$ is a numerical semigroup.*

Proof. We always have $0 \in H_f(P)$. Further, for every $f, g \in \mathcal{M}(G)$, we have $\Delta(f+g) = \Delta f + \Delta g$, so $H_f(P)$ is closed under addition. Moreover, if G is simple, from Theorem 2.3.4 we have $H_r(P) \subseteq H_f(P)$, and from Lemma 2.2.5 it follows $|\mathbb{N} \setminus H_f(P)| \leq |\mathbb{N} \setminus H_r(P)| = g(G)$, where $g(G)$ is the genus of G. Therefore $H_f(P)$ is a numerical semigroup. □

When the graph G is not clear from the context, we denote the functional Weierstrass set by $H_f^G(P)$. For two subsets $A, B \subseteq \mathbb{N}$, we define

$$A + B = \{a + b : a \in A, b \in B\}.$$

Proposition 2.4.2. *Let G_1 and G_2 be two graphs, and let G be the graph obtained from G_1 and G_2 by the vertex gluing of $P_1 \in V(G_1)$ and $P_2 \in V(G_2)$, and denote by $P \in V(G)$ the identified vertex in G. Then*

$$H_f^G(P) = H_f^{G_1}(P_1) + H_f^{G_2}(P_2).$$

Proof. We will consider G_1 and G_2 as subgraphs of G. For simplicity, set $S = H_f^G(P)$ and $S_i = H_f^{G_i}(P_i)$ for $i \in \{1, 2\}$. Let $x \in S_1$, then there exists $f \in \mathcal{M}(G_1)$ such that $\Delta(f) = D - xP_1$ for some effective divisor $D \geq 0$. Consider the extension f' of f to G by setting $f'(v) = f(P)$ for all $v \in V(G_2) \setminus \{P\}$. Then $\Delta(f') = \Delta(f)$ and $x \in S$. This proves $S_1 \subseteq S$. Similarly we obtain $S_2 \subseteq S$, thus $S_1 + S_2 \subseteq S$ since S is closed under addition.

On the other hand, let $x \in S$. Then there exists $f \in \mathcal{M}(G)$ such that $\Delta(f) = D - xP$ for some $D \geq 0$. Substituting f with $f + a$ for some constant $a \in \mathbb{Z}$ if necessary, we can assume that $f(P) = 0$. For $i \in \{1, 2\}$, define

$$f_i \in \mathcal{M}(G_i) \quad f_i(v) = f(v) \quad \forall v \in V(G_i) \subseteq V(G).$$

Since $f(P) = 0$, we have $\Delta_v(f_i) = \Delta_v(f) \geq 0$ for all $v \in V(G_i) \setminus \{P\} \subseteq V(G)$, with $i \in \{1, 2\}$. Since every principal divisor has degree zero, we have

$$\Delta(f_1) = D_1 - x_1 P, \quad \Delta(f_2) = D_2 - x_2 P$$

for some $x_1, x_2 \in \mathbb{N}$ and some effective divisor $D_i \geq 0$ on G_i for $i \in \{1, 2\}$. From the definition we have $f = f_1 + f_2$, therefore $\Delta(f) = \Delta(f_1) + \Delta(f_2)$, hence $x = x_1 + x_2 \in S_1 + S_2$. □

Corollary 2.4.3. *For every additive submonoid M of $\mathbb{N}$ there exists a graph G such that $M = H_f(P)$ for some $P \in V(G)$.*

Proof. From [100, Lemma 2.3], every additive submonoid of $\mathbb{N}$ is finitely generated, so suppose that $M = \langle n_1, \dots, n_e \rangle$. Now let G be the graph with vertices $V(G) = \{P, P_1, \dots, P_e\}$ where the vertex P has n_i edges connected to the vertex P_i for every $i \in \{1, \dots, e\}$. From Remark 2.3.5 and Proposition 2.4.2 it follows that $H_f(P) = n_1\mathbb{N} + \cdots + n_e\mathbb{N} = M$. □

Corollary 2.4.4. *For every numerical semigroup S there exists a simple graph G such that $S = H_f(P)$ for some $P \in V(G)$.*

Proof. Suppose that $S = \langle n_1, \dots, n_e \rangle$. Set $m = \max(\mathbb{N} \setminus S) + 2$ and consider the complete bipartite graphs $K_{m,n_1}, \dots, K_{m,n_e}$. Fix a vertex of degree n_i in each graph and construct the graph G by identifying these vertices, recursively applying the vertex gluing. Denote with P the identified vertex in G. From Proposition 2.4.2 and Lemma 2.3.9 we obtain $H_f(P) = S$. □

Using Lemma 2.4.1 and Corollary 2.4.3 and 2.4.4 we now state the main result of this section.

Theorem 2.4.5. *The functional Weierstrass sets of graphs (resp. simple graphs) are precisely the additive submonoids of $\mathbb{N}$ (resp. numerical semigroups).*

We close the section by calculating the multiplicity of the functional Weierstrass set of a simple graph. Recall that the multiplicity of a numerical semigroup S is the integer $m(S) = \min(S \setminus \{0\})$. Let G be a simple graph and fix a vertex $P \in V(G)$. Denote with $G - P$ the graph obtained from G by removing the vertex P and its adjacent edges.

Lemma 2.4.6. *[70, Theorem 2.10] Suppose that $G - P$ is connected. Then $m(H_f(P)) = \deg(P)$.*

Proposition 2.4.7. *Let $G_1, \dots, G_m$ be the connected components of $G - P$, and let $\deg_{G_i} P$ be the number of edges incident with P in G_i. Then*

$$m(H_f(P)) = \min\{\deg_{G_i} P : i \in \{1, \dots, m\}\}.$$

Proof. Let C_i be the graph obtained from G_i by adding the vertex P and the edges of G incident with P in G_i. The graph G can be seen as the vertex gluing of the graphs C_i along P. Now it is enough to apply Proposition 2.4.2 and Lemma 2.4.6. □

2.5 Rank Weierstrass sets

Let G be a graph and fix a vertex $P \in V(G)$. Define the function $\lambda_P : \mathbb{N} \to \mathbb{N}$ by

$$\lambda_P(k) = \min\{n \in \mathbb{N} : r(nP) = k\}.$$

Note that the function λ_P is an order preserving bijection between $\mathbb{N}$ and $H_r(P)$. Thus, λ_P completely determines $H_r(P)$ and vice versa. We will write λ_P^G when the graph G is not clear from the context.

Proposition 2.5.1. *Let G_1 and G_2 be two graphs and fix $P_i \in V(G_i)$ for $i \in \{1,2\}$. Let G be the vertex gluing of P_1 and P_2, and let P be the identified vertex. Then*

$$\lambda_P^G(k) = \max\left\{\lambda_{P_1}^{G_1}(k_1) + \lambda_{P_2}^{G_2}(k_2) : k_1 + k_2 = k\right\}.$$

Proof. We will consider G_1 and G_2 as subgraphs of G. First, note that every divisor $E \in \mathrm{Div}_+^k(G)$ can be decomposed as the sum $E = E_1 + E_2$ where $E_i \in \mathrm{Div}_+^{k_i}(G_i)$ for $i \in \{1,2\}$ with $k_1 + k_2 = k$. Further, if Δf is a principal divisor in G, without loss of generality we can assume that $f(P) = 0$, so that $f = f_1 + f_2$ with $f_1 = 0$ in G_2 and $f_2 = 0$ in G_1. It follows that $\Delta f = \Delta f_1 + \Delta f_2$, in other words any principal divisor in G is the sum of two principal divisors in G_1 and G_2 respectively.

Claim. *Let $n, k \in \mathbb{N}$, the following statements are equivalent:*

1. $|nP - E| \neq \emptyset$ *for every* $E \in \mathrm{Div}_+^k(G)$,
2. $n \geq \lambda_{P_1}^{G_1}(k_1) + \lambda_{P_2}^{G_2}(k_2)$ *for every* $k_1 + k_2 = k$.

Proof of claim. First of all, set $n_i = \lambda_{P_i}^{G_i}(k_i)$ for $i \in \{1,2\}$.

1) $\Rightarrow$ 2) Let $k_1, k_2 \in \mathbb{N}$ such that $k_1 + k_2 = k$. By the definition of n_i, there exists $E_i \in \mathrm{Div}_+^{k_i}(G_i)$ such that $|(n_i - 1)P - E_i| = \emptyset$ for $i \in \{1,2\}$. Set $E = E_1 + E_2 \in \mathrm{Div}_+^k(G)$. By hypothesis we have

$$nP - E + \Delta f = nP + (\Delta f_1 - E_1) + (\Delta f_2 - E_2) \geq 0,$$

for some $f \in \mathcal{M}(G)$, with $f = f_1 + f_2$ as described above. Assume by contradiction $n < n_1 + n_2$, this means that, for some $i \in \{1,2\}$, we have $(n_i - 1)P - E_i + \Delta f_i \geq 0$, that is $|(n_i - 1)P - E_i| \neq \emptyset$, contradiction.

2) $\Rightarrow$ 1) Let $E \in \mathrm{Div}_+^k(G)$, then $E = E_1 + E_2$ with $E_i \in \mathrm{Div}_+^{k_i}(G_i)$ for $i \in \{1,2\}$ and $k_1 + k_2 = k$. By the definition of n_i, there exists $f_i \in \mathcal{M}(G_i)$ such that $n_i P - E_i + \Delta f_i \geq 0$ for $i \in \{1,2\}$. Without loss of generality, we can assume that $f_1(P) = f_2(P) = 0$, set $f = f_1 + f_2$, we have

$$nP - E + \Delta f \geq \sum_{i \in \{1,2\}} (n_i P - E_i + \Delta f_i) \geq 0$$

that is $|nP - E| \neq \emptyset$. $\square$

Now write

$$\begin{aligned}
\lambda_P^G(k) &= \min\{n \in \mathbb{N} : r(nP) \geq k\} \\
&= \min\{n \in \mathbb{N} : |nP - E| \neq \emptyset, \, \forall E \in \mathrm{Div}_+^k(G)\} \\
&= \min\{n \in \mathbb{N} : n \geq \lambda_{P_1}^{G_1}(k_1) + \lambda_{P_2}^{G_2}(k_2), \text{ for every } k_1 + k_2 = k\} \\
&= \max\left\{\lambda_{P_1}^{G_1}(k_1) + \lambda_{P_2}^{G_2}(k_2) : k_1 + k_2 = k\right\}. \qquad \square
\end{aligned}$$

Remark 2.5.2. We note that the notion of rank Weierstrass set implicitly appears in [96, Section 3.2]. In fact, the so called *Weierstrass partition* of the zero divisor at a marked point P encodes the information of the rank Weierstrass set $H_r(P)$ (see [96, Definition 3.11]). Further, we note that one of the main techniques used in [96] to study the behaviour of Weierstrass partitions is the (analogue of) vertex gluing of an arbitrary metric graph with a cycle.

Theorem 2.5.3. *Let $e_1 \geq e_2 \geq \cdots \geq e_n \geq 0$ be integers and set $s_i = \sum_{j=1}^{i} e_j$. There exists a simple graph G with a vertex $P \in V(G)$ such that*

$$H_r(P) = \{0, s_1, s_2, \ldots, s_{n-1}\} \cup (s_n + \mathbb{N}).$$

Proof. We proceed by induction on n. For the base case $n = 1$, by Corollary 2.3.10 it is enough to consider the graph K_{2,e_1} and a vertex P of degree e_1. Now assume that the theorem is true for $n-1$, and let G' be a graph with a vertex P_1 such that

$$H_r^{G'}(P_1) = \{0, s_1, \ldots, s_{n-1}\} \cup (s_{n-1} + \mathbb{N}).$$

Consider the graph K_{2,e_n} and fix a vertex P_2 of degree e_n. Let G be the vertex gluing of P_1 and P_2. From Corollary 2.3.10 we have

$$H_r^{K_{2,e_n}}(P_2) = \{0\} \cup (e_n + \mathbb{N}).$$

Now apply Proposition 2.5.1 to the vertex gluing G of P_1 and P_2. $\square$

Remark 2.5.4. In the proof of Theorem 2.5.3 we glued together complete bipartite graphs K_{2,e_i} along vertices of degree e_i. However, we could have used any graph of genus $e_i - 1$ with a fixed non-Weierstrass point, i.e. with a fixed vertex in which the Weierstrass set is $\mathbb{N} \setminus \{1, \ldots, e_i - 1\}$.

Remark 2.5.5. A numerical semigroup S is *Arf* if for every $x, y, z \in S$ with $x \geq y \geq z$ we have $x + y - z \in S$. From Theorem 2.5.3 it follows that every Arf numerical semigroup is the rank Weierstrass set of some graph. In fact, it is enough to choose the sequence

$e_1 \geq \cdots \geq e_n$ to be the multiplicity sequence of the given Arf numerical semigroup. See [9, Section 2] for more information about Arf numerical semigroups and their multiplicity sequence.

Theorem 2.5.3 can be used to construct families of graphs with rank Weierstrass set that is not a semigroup. We now describe an example of such a graph.

Example 2.5.6. Let $n = 3$ and $(e_1, e_2, e_3) = (3, 2, 2)$. Following the idea in the proof of Theorem 2.5.3, we consider the graph G (Figure 2.1) obtained as the vertex gluing of $K_{2,3}$ and two copies of $K_{2,2}$. Let $P \in V(G)$ be the identified vertex of degree 7. We have

$$H_r(P) = \{0, 3, 5, 7\} \cup (8 + \mathbb{N}).$$

Note that $H_r(P)$ is not a semigroup, since $3 \in H_r(P)$, but $3 + 3 = 6 \notin H_r(P)$. $\diamondsuit$

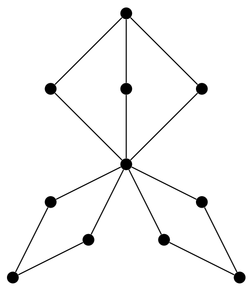


Figure 2.1

Theorem 2.5.3 gives a sufficient condition for a subset of $\mathbb{N}$ to be the rank Weierstrass set of some graph. We now provide an easy necessary condition.

Proposition 2.5.7. *Let G be a graph and fix a vertex $P \in V(G)$. For every $n, k \in \mathbb{N}$ we have*

$$\left|H_r(P) \cap [1, nk]\right| \geq k\left|H_r(P) \cap [1, n]\right|.$$

Proof. From the definition we have $r(nP) = \left|H_r(P) \cap [1, n]\right|$, further from Lemma 2.2.1 $r(nkP) \geq k\, r(nP)$. $\square$

Question 2.5.8. Can we characterize the cofinite subsets $H \subseteq \mathbb{N}$ that are the rank Weierstrass set of some graph?

In [8] the notion of harmonic morphism between graphs is studied. It is a discrete analogue of morphisms of curves. In particular, in this context it makes sense to talk about hyperelliptic graphs and double covers.

Classically, we know that a curve X is hyperelliptic if and only if there exists $P \in X$ such that $2 \in H(P)$. An analogous fact is proved in [109, Theorem A]: a curve X

of genus $g \geq 6\gamma + 4$ is a double cover of a curve of genus $\gamma \geq 1$ if and only if there exists $P \in X$ such that $H(P)$ has γ even gaps.

Question 2.5.9. Can we find an analogue of [109, Theorem A] for graphs?

Chapter 3

Tropical linear degenerate flag varieties

This chapter is a readaptation of a joint work with Victoria Schleis.

3.1 Introduction

The Grassmannian $G(r;n)$ parametrizes r-dimensional linear subspaces U of an n-dimensional K-vector space V, and can be embedded in the projective space $\mathbb{P}^{\binom{n}{r}-1}$ with equations given by the Grassmann-Plücker relations.

A generalization of the Grassmannian is the flag variety $\mathrm{Fl}(\mathbf{r};n)$ parametrizing flags of linear spaces of rank $\mathbf{r}=(r_1,\ldots,r_k)$, that is, sequences of linear subspaces $(U_1,\ldots,U_k)$ of V with $\dim(U_i)=r_i$ such that $U_i\subseteq U_{i+1}$ for all $i\in\{1,\ldots,k-1\}$. Similarly, a flag variety can be embedded in a product of projective spaces and the equations of this embedding are given by the incidence Plücker relations in addition to the Grassmann-Plücker relations.

Flag varieties are further generalized by *linear degenerate flag varieties* parametrizing *linear degenerate flags* of linear subspaces. These are defined as follows: fix a sequence f_* of linear maps $f_i: V\to V$ for $i\in\{1,\ldots,k-1\}$ and a rank vector $\mathbf{r}=(r_1,\ldots,r_k)$. An f_*-linear degenerate flag is a sequence of linear subspaces $(U_1,\ldots,U_k)$ of V with $\dim(U_i)=r_i$ and such that $f_i(U_i)\subseteq U_{i+1}$ for every $i\in\{1,\ldots,k-1\}$. The flag variety is the f_*-linear degenerate flag variety where all the f_i are equal to the identity. Further, every f_*-linear degenerate flag variety can be given as a sequence of projections, using the $GL(V)$ action on V for each f_i, (see [49, Lemma 2.6]). Hence, in this paper we will restrict to projections.

The tropicalization $\overline{\mathrm{trop}}(G(r;n))$ of the Grassmannian (thought inside the *tropical projective space* $\mathbb{P}(\mathbb{T}^{\binom{n}{r}})$, see Section 3.2.1 for more details) parametrizes *realizable* valuated matroids of rank r on n elements, or equivalently *realizable* tropical linear spaces (i.e. tropicalizations of linear spaces) of dimension r in $\mathbb{P}(\mathbb{T}^n)$. The object parametrizing all valuated matroids of rank r on n elements, or equivalently all tropical linear spaces of dimension r inside $\mathbb{P}(\mathbb{T}^n)$, is a tropical prevariety $\mathrm{Dr}(r,n)$ called the *Dressian* (see [106]). Its equations are the tropical Grassmann-Plücker relations, and we have $\overline{\mathrm{trop}}(G(r;n)) \subseteq \mathrm{Dr}(r,n)$, although in general this inclusion might be strict.

Similarly, the tropicalization of the flag variety $\overline{\mathrm{trop}}(\mathrm{Fl}(\mathbf{r};n))$ parametrizes realizable valuated flag matroids, or equivalently realizable flags of tropical linear spaces (see Section 3.2 for precise definitions).

In [19], Brandt, Eur and Zhang defined the *flag Dressian* $\mathrm{FlDr}(\mathbf{r};n)$, a tropical prevariety with equations given by the incidence Plücker relations and the Grassmann-Plücker relations. They proved that $\mathrm{FlDr}(\mathbf{r};n)$ parametrizes valuated flag matroids or equivalently flags of tropical linear spaces (see Theorem 3.2.9).

In this paper, we define the *linear degenerate flag Dressian* $\mathrm{LFlDr}(\mathbf{r},\mathbf{S};n)$ of rank $\mathbf{r}=(r_1,\dots,r_k)$ and degeneration type $\mathbf{S}=(S_1,\dots,S_{k-1})$. Here the r_i are the dimensions of the linear spaces of the linear degenerate flags, whereas the S_i are the set of indices corresponding to the projections pr_{S_i} defined by: $\mathrm{pr}_{S_i}(e_j)=0$ if $j\in S_i$ and $\mathrm{pr}_{S_i}(e_j)=e_j$ otherwise. The variety $\mathrm{LFlDr}(\mathbf{r},\mathbf{S};n)$ is defined as the tropical (pre)variety with equations given by the linear degenerate incidence Plücker relations and the Grassmann-Plücker relations. The following is our main result, which is a generalization of the work of Brandt, Eur and Zhang [19].

Theorem A. *Let $\boldsymbol{\mu}=(\mu_1,\dots,\mu_k)$ be a sequence of valuated matroids. The following statements are equivalent:*

(a) $\boldsymbol{\mu}\in\mathrm{LFlDr}(\mathbf{r},\mathbf{S};n)$*;*

(b) $\boldsymbol{\mu}$ *is a linear degenerate valuated flag matroid;*

(c) $\mathrm{pr}_{S_i}^{\mathrm{trop}}\big(\overline{\mathrm{trop}}(\mu_i)\big)\subseteq\overline{\mathrm{trop}}(\mu_{i+1})$ *for all* $i\in\{1,\dots,k-1\}$*;*

(d) every projection $\mathrm{pr}_{S_i}:\mu_{i+1}\to\mu_i$ *is a morphism of valuated matroids.*

The first statement in the above theorem expresses the fact that the valuations of the bases of the valuated matroids μ_i satisfy the corresponding Plücker relations. The second one expresses the fact that the matroids μ_i form certain valuated matroids quotients. The third statement is about containments of certain projections of the tropical linear spaces of the matroids μ_i. The last statement is a recast of the second one in terms of morphisms of valuated matroids. See Section 3.3 for precise definitions.

Figure 3.1 represents two tropical flags. The flag on the left is a (classical) tropical flag, consisting of a point (in yellow) contained in a tropical line (in red), contained in a tropical plane (in blue). The flag on the right is a linear degenerate tropical flag: while the tropical line is yet contained in the tropical plane, only a projection of the point is contained in the tropical line.

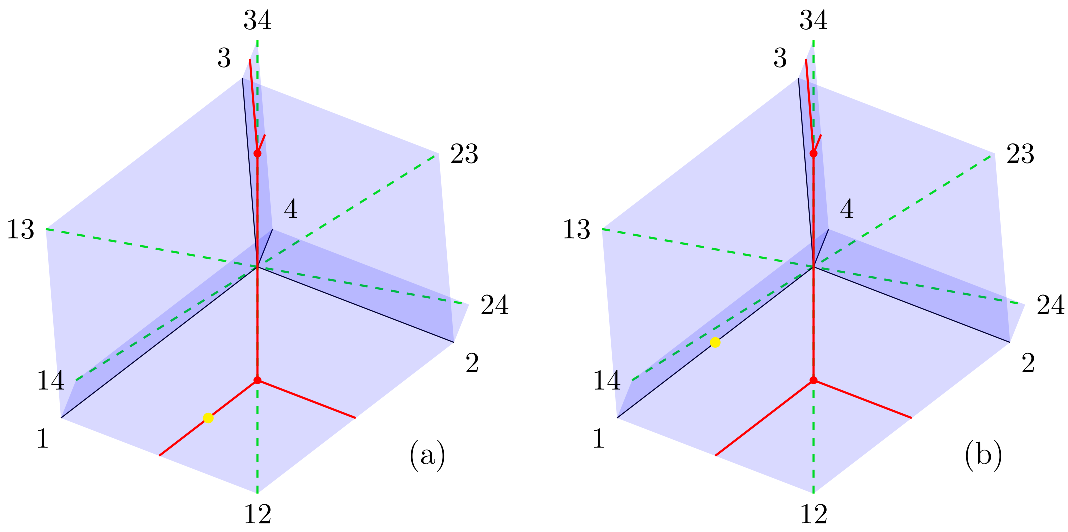


Figure 3.1: (a): A tropical flag in $\mathrm{trop}\left(\mathrm{Fl}((1,2,3);4)\right)$. (b): A tropical linear degenerate flag in $\mathrm{trop}\left(\mathrm{LFl}\left((1,2,3),(\{1\},\emptyset);4\right)\right)$. Both are made of a yellow point, a red tropical line, and a blue tropical plane. The additional subdivision given by the green dashed rays on the tropical plane is useful for describing the (linear degenerate) tropical flag varieties, see Examples 3.4.1 and 3.4.2.

The two extreme cases of linear degenerate flag varieties are flag varieties (by setting all $S_i = \emptyset$) and products of Grassmannians (by setting all $S_i = \{1,\dots,n\}$). Hence, linear degenerate flag varieties bridge the gap between products of Grassmannians and flag varieties, see Section 3.4 for more details.

An analogous statement holds for the realizable case. We obtain the following correspondence:

Theorem B. *Let $\boldsymbol{\mu} = (\mu_1,\dots,\mu_k)$ be a sequence of* realizable *valuated matroids. The following statements are equivalent:*

(a) $\boldsymbol{\mu} \in \overline{\mathrm{trop}}(\mathrm{LFl}(\mathbf{r},\mathbf{S};n))$*;*

(b) $\boldsymbol{\mu}$ *is a* realizable *linear degenerate valuated flag matroid;*

(c) $\mathrm{pr}^{\mathrm{trop}}_{S_i}\left(\overline{\mathrm{trop}}(\mu_i)\right) \subseteq \overline{\mathrm{trop}}(\mu_{i+1})$ *for all* $i \in \{1,\dots,k-1\}$ *and there exist realizations* $L_1,\dots,L_k$ *of* $\mu_1,\dots,\mu_k$ *such that* $\mathrm{pr}_{S_i}(L_i) \subseteq L_{i+1}$ *for all* $i \in \{1,\dots,k-1\}$.

(d) every projection $\mathrm{pr}_{S_i} : \mu_{i+1} \to \mu_i$ *is a* realizable *morphism of valuated matroids, and they can be realized* simultaneously, *using the same realization of* μ_i*, for all* $i \in \{1,\dots,k\}$.

Linear degenerate flag varieties are of interest in representation theory. They were studied in [23, 24, 43, 44, 45, 97]. It was observed in [25] that linear degenerate flag varieties are isomorphic to quiver Grassmannians for type A quivers. Quiver Grassmannians first appeared in [29, 102]. They are varieties parametrizing subrepresentations of quiver representations. Notably, every projective variety is isomorphic to a quiver Grassmannian [98]. In addition, every quiver Grassmannian can be naturally embedded in a product of Grassmannians, and the equations of this embedding were described in [82].

A first step towards *tropical flag varieties* was made in [59] by Haque, showing that the flag Dressian parametrizes flags of tropical linear spaces. The tropicalization of (complete) flag varieties of length 4 and 5 were computed in [18]. In a related work [39], tropical flag varieties are linked to PBW-degenerations. Further, in [67], in order to combinatorially describe tropicalized Fano schemes, previously introduced in [79], the authors studied the space of valuated flag matroids (μ_1, μ_2) of rank $(r, r+1)$ where μ_2 is fixed. A generalization of valuated flag matroids to tracts has been made in [66]. Positivity for tropical flag Dressians has been studied in [5, 11, 12, 69, 105].

This paper is structured as follows. In Section 3.2 we review the needed background knowledge and fix our notation. In Section 3.3 we define the linear degenerate flag Dressian and prove our main results, Theorem A and Theorem B. Finally, in Section 3.4 we give some examples, first results on relationships of linear degenerate flag varieties and Dressians with similar rank but different degeneration set and discuss some possible applications and future work.

3.2 Preliminaries

3.2.1 Tropical Geometry

In this section, we review the basics of tropical geometry. Our main reference is [85]. We follow the min-convention for all tropical and matroidal operations.

Set $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ and define $a \oplus b = \min\{a, b\}$ and $a \odot b = a + b$ for every $a, b \in \mathbb{T}$. Then $(\mathbb{T}, \oplus, \odot)$ is a semifield, called the *tropical semifield*. The *tropical projective space* is $\mathbb{P}(\mathbb{T}^n) = (\mathbb{T}^n \setminus \{(\infty, \dots, \infty)\})/\mathbb{R}\mathbf{1} = (\mathbb{T}^n \setminus \{(\infty, \dots, \infty)\})/\sim$. Here we are quotienting by the equivalence relation defined by: $\mathbf{u} \sim \mathbf{v}$ if $\mathbf{u} = \mathbf{v} + c\mathbf{1}$ for some $c \in \mathbb{R}$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. A *tropical polynomial* is an element of the semiring $\mathbb{T}[x_0, \dots, x_n]$ in the variables $x_0, \dots, x_n$ with coefficients in $\mathbb{T}$. The *tropical hypersurface* of a tropical polynomial $F = \bigoplus_{u\in\mathbb{N}^{n+1}} c_u \odot x^u \in \mathbb{T}[x_0,\dots,x_n]$ is

$$V(F) = \left\{ x \in \mathbb{P}(\mathbb{T}^n) : \text{the minimum in} \left\{ c_u + \sum_{i=0}^{n} u_i \cdot x_i \right\}_{u\in\mathbb{N}^{n+1}} \text{is achieved at least twice} \right\},$$

where whenever $\min_{u\in\mathbb{N}^{n+1}}\{c_u + \sum_{i=0}^{n} u_i \cdot x_i\} = \infty$, by convention the minimum is achieved at least twice, even if the expression is a tropical monomial. The *tropical variety* of an ideal of tropical polynomials $J \subseteq \mathbb{T}[x_0,\dots,x_n]$ is defined by

$$V(J) = \bigcap_{F\in J} V(F) \subseteq \mathbb{P}(\mathbb{T}^n).$$

In the following, let K be a field with valuation $\mathrm{val} : K \to \mathbb{T}$. The *tropicalization* of a polynomial $f = \sum_{u\in\mathbb{N}^{n+1}} a_u x^u \in K[x_0,\dots,x_n]$ is the tropical polynomial

$$\mathrm{trop}(f) = \bigoplus_{u\in\mathbb{N}^{n+1}} \mathrm{val}(a_u) \odot x^u \in \mathbb{T}[x_0,\dots,x_n].$$

The *tropicalization* $\mathrm{trop}(I)$ of an ideal $I \subseteq K[x_0,\dots,x_n]$ is the ideal of tropical polynomials generated by the tropicalizations of all polynomials in I:

$$\mathrm{trop}(I) = \langle \mathrm{trop}(f) : f \in I \rangle \subseteq \mathbb{T}[x_0,\dots,x_n].$$

The *tropicalization* $\overline{\mathrm{trop}}(X)$ of a subvariety $X \subseteq \mathbb{P}^n$, when the base field K is algebraically closed with a non-trivial valuation, is defined by

$$\overline{\mathrm{trop}}(X) = \overline{\{(\mathrm{val}(x_0),\dots,\mathrm{val}(x_n)) \in \mathbb{P}(\mathbb{T}^n) : [x_0 : \cdots : x_n] \in X\}},$$

where the closure is with respect to the Euclidean topology induced on $\mathbb{P}(\mathbb{T}^n)$.

Now we have two possible ways of constructing a tropical variety from a homogeneous ideal of polynomials $I \subseteq K[x_0,\dots,x_n]$: we can first tropicalize the ideal, and then take its tropical variety $V(\mathrm{trop}(I))$, or we can consider the affine variety $V(I)$ and tropicalize it to obtain $\overline{\mathrm{trop}}(V(I))$. The fundamental theorem of tropical geometry assures us that, over algebraically closed fields with nontrivial valuation, these two operations yield the same result, i.e. that $\overline{\mathrm{trop}}(V(I)) = V(\mathrm{trop}(I))$, see [85, Theorem 6.2.15].

We use the notation $\overline{\mathrm{trop}}(X)$ for the tropicalization of a subvariety $X \subseteq \mathbb{P}^n$ and denote by $\mathrm{trop}(X)$ the intersection $\overline{\mathrm{trop}}(X) \cap (\mathbb{R}^n/\mathbb{R}\mathbf{1})$, which is the tropicalization inside $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ of the intersection $X \cap T^n$, where $T^n \simeq (K \setminus \{0\})^n$ is the algebraic torus inside $\mathbb{P}^n$. That is, $\overline{\mathrm{trop}}(X)$ might contain points in which some of the coordinates are ∞. For a thorough description of this extension of tropical varieties, we refer to [19, Section 2] and [85, Section 6.2].

Further, tropical varieties have a nice polyhedral structure: for an irreducible subvariety $V(I)$ of dimension d in the torus T^n, the tropical variety $\mathrm{trop}(V(I))$ is the support of a d-dimensional, rational, balanced polyhedral complex that is connected through codimension one, see [85, Theorem 3.3.5] for more details.

3.2.2 Valuated matroids and tropical linear spaces

Throughout, we will assume that the reader is familiar with basic matroid theory, see [95] and [111]. We denote $[n]$ for the set $\{1, 2, \ldots, n\}$ and $\binom{[n]}{r}$ for the family of subsets of $[n]$ of cardinality r.

Definition 3.2.1. A *valuated matroid* of rank r on the ground set $[n]$ is a function $\nu : \binom{[n]}{r} \to \mathbb{T}$ such that: $\nu(B) \neq \infty$ for some $B \in \binom{[n]}{r}$, and for all $I, J \in \binom{[n]}{r}$ and $i \in I \setminus J$ there exists $j \in J \setminus I$ satisfying

$$\nu(I) + \nu(J) \geq \nu((I \setminus i) \cup j) + \nu((J \setminus j) \cup i).$$

Two rank r valuated matroids μ and ν on the common ground set $[n]$ are *equivalent* if there exists $a \in \mathbb{R}$ such that $\mu(B) = \nu(B) + a$ for every $B \in \binom{[n]}{r}$. In other words, every equivalence class of a valuated matroid $\nu : \binom{[n]}{r} \to \mathbb{T}$ can be seen as a point in $\mathbb{P}(\mathbb{T}^{\binom{n}{r}})$. Throughout, we will regard two equivalent valuated matroids as being the same, and consider valuated matroids only up to equivalence.

If $\nu : \binom{[n]}{r} \to \mathbb{T}$ is a valuated matroid, then $\{B \in \binom{[n]}{r} : \nu(B) \neq \infty\}$ is a collection of bases of a matroid N, called the *underlying matroid* of ν.

Definition 3.2.2. Let M and N be two matroids over the same ground set $[n]$. We say that M is a *matroid quotient* of N, denoted $M \twoheadleftarrow N$, if every flat of M is a flat of N.

Definition 3.2.3 ([19, Definition 4.2.2])**.** Let μ and ν be two valuated matroids on the ground set $[n]$ of rank $r \leq s$ respectively. We say that μ is a *valuated matroid quotient* of ν, denoted $\mu \twoheadleftarrow \nu$, if for every $I \in \binom{[n]}{r}$, $J \in \binom{[n]}{s}$ and $i \in I \setminus J$, there exists $j \in J \setminus I$ such that

$$\mu(I) + \nu(J) \geq \mu(I \cup j \setminus i) + \nu(J \cup i \setminus j).$$

If $\mu \twoheadleftarrow \nu$ is a valuated matroid quotient, and M and N are the underlying matroids of μ and ν respectively, then $M \twoheadleftarrow N$.

Definition 3.2.4. A sequence of valuated matroids $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ on a common ground set $[n]$, is a *valuated flag matroid* if $\mu_i \twoheadleftarrow \mu_j$ for every $1 \leq i \leq j \leq k$. Analogously, a sequence of matroids $\mathbf{M} = (M_1, \ldots, M_k)$ on $[n]$ is a *flag matroid* if $M_i \twoheadleftarrow M_j$ for every $1 \leq i \leq j \leq n$.

Let K be a field with valuation $\mathrm{val} : K \to \mathbb{T}$. Let L be an r-dimensional vector subspace of K^n given as the row span of a matrix A. We denote by (p_I) the *Plücker coordinates* of L, where p_I is defined as the minor of A indexed by $I \in \binom{[n]}{r}$. Then, the function $\mu(L) : \binom{[n]}{r} \to \mathbb{T}$ defined by $I \mapsto \mathrm{val}(p_I)$ is a valuated matroid satisfying Definition 3.2.1, and we denote by $M(L)$ its underlying matroid. The valuated matroid $\mu(L)$ is well defined only up to equivalence, since different choices of the matrix A will give rise to equivalent valuated matroids. Matroids arising in this way are called *realizable* (over K). As in [19, Example 4.1.2], if $L_1 \subseteq L_2$ are two linear subspaces of K^n, then we have $\mu(L_1) \twoheadleftarrow \mu(L_2)$. Matroid quotients arising in this way are called *realizable* (over K). Note that a matroid quotient $M \twoheadleftarrow N$ of two realizable matroids is not necessarily realizable (see [13, §1.7.5, Example 7]).

Definition 3.2.5. Let μ be a valuated matroid of rank r on $[n]$. For each $I \in \binom{[n]}{r+1}$ define an element $C_\mu(I) \in \mathbb{T}^n$ by

$$C_\mu(I)_i = \begin{cases} \mu(I \setminus i) & i \in I, \\ \infty & i \notin I. \end{cases}$$

The set of *valuated circuits* $\mathcal{C}(\mu)$ of μ is defined as the image in $\mathbb{P}(\mathbb{T}^n)$ of the following set:

$$\left\{ C_\mu(I) : I \in \binom{[n]}{r+1} \right\} \setminus \{(\infty, \ldots, \infty)\}.$$

A *cycle* of a matroid is a union of circuits. A *vector* (or *valuated cycle*) of μ is any element of $\mathbb{P}(\mathbb{T}^n)$ (tropically) generated by the valuated circuits. More explicitly, the family of vectors is

$$\mathcal{V}(\mu) = \left\{ \bigoplus_{C \in \mathcal{C}(\mu)} \lambda_C \odot C : \lambda_C \in \mathbb{T}, \lambda_C \neq \infty \right\}.$$

For more information about vectors of a valuated matroid see [83, Section 2.1].

Definition 3.2.6. Let μ be a valuated matroid on $[n]$. The *tropical linear space* of μ is the tropical variety

$$\overline{\mathrm{trop}}(\mu) = \bigcap_{C \in \mathcal{C}(\mu)} V\left(\bigoplus_{i \in [n]} C_i \odot x_i \right) \subseteq \mathbb{P}(\mathbb{T}^n).$$

3.2.3 Plücker relations and the Flag Dressian

In the following, let K be a field with valuation $\mathrm{val} : K \to \mathbb{T}$.

Definition 3.2.7. Let $r \le s \le n$ be nonnegative integers. The *incidence Plücker relations* are the polynomials in the variables $\{p_I : I \in \binom{[n]}{r}\} \cup \{p_J : J \in \binom{[n]}{s}\}$ with coefficients in K:

$$\mathscr{P}_{r,s;n} = \left\{ \sum_{j \in J \setminus I} \mathrm{sgn}(j; I, J) p_{I \cup j} p_{J \setminus j} : I \in \binom{[n]}{r-1}, J \in \binom{[n]}{s+1} \right\}$$

where $\mathrm{sgn}(j; I, J) = (-1)^{\#\{j' \in J : j < j'\} + \#\{i \in I : i > j\}}$. The tropicalization of the incidence Plücker relations are denoted by $\mathscr{P}^{\mathrm{trop}}_{r,s;n}$. The *Grassmann-Plücker relations* are the polynomials $\mathscr{P}_{r;n} := \mathscr{P}_{r,r;n}$, and their tropicalization are denoted by $\mathscr{P}^{\mathrm{trop}}_{r;n}$. They are the equations of the Plücker embedding of the Grassmannian $G(r; n)$.

The incidence-Plücker relations are, combined with the Grassmann-Plücker relations, the equations of flag varieties, defined as follows. Let $r_1 \le \cdots \le r_k \le n$ be nonnegative integers, and set $\mathbf{r} = (r_1, \dots, r_k)$. The *flag variety* is the following subvariety of $\mathbb{P}^{\binom{n}{r_1}-1} \times \cdots \times \mathbb{P}^{\binom{n}{r_k}-1}$

$$\mathrm{Fl}(\mathbf{r}; n) = V\big(\{\mathscr{P}_{r_i;n}\}_{1 \le i \le k} \cup \{\mathscr{P}_{r_i,r_j;n}\}_{1 \le i < j \le k}\big).$$

Remark 3.2.8. Grassmannians and flag varieties have two respective tropical analogues. *Tropical Grassmannians* $\mathrm{trop}(G(r; n))$ and *tropical flag varieties* $\mathrm{trop}(\mathrm{Fl}(\mathbf{r}; n))$ are tropicalizations of their classical analogues, as in Section 3.2.1. These *tropical varieties* parametrize *tropicalized* objects: $\mathrm{trop}(G(r; n))$ parametrizes tropicalizations of linear subspaces of K^n of dimension r and $\mathrm{trop}(\mathrm{Fl}(\mathbf{r}; n))$ parametrizes realizable tropical flags $\overline{\mathrm{trop}}(L_1) \subseteq \cdots \subseteq \overline{\mathrm{trop}}(L_k)$ where $L_1 \subseteq \cdots \subseteq L_k$ is a flag of subspaces of K^n satisfying $\dim L_i = r_i$. *Dressians* $\mathrm{Dr}(r; n)$ and *flag Dressians* $\mathrm{FlDr}(\mathbf{r}; n)$ are the intersections of the tropical hypersurfaces given by their respective Plücker relations. They are *tropical prevarieties* and parametrize *tropical* objects. In general, (flag) Dressians and tropical Grassmannians are different polyhedral complexes. The Dressian $\mathrm{Dr}(r; n)$ parametrizes (not necessarily realizable) tropical linear spaces of rank r in $\mathbb{P}(\mathbb{T}^n)$ as in Definition 3.2.6. Flag Dressians parametrize (not necessarily realizable) flags of tropical linear spaces:

Theorem 3.2.9 ([59, Theorem 1],[19, Theorem A])**.** *Let $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k)$ be a sequence of valuated matroids on common ground set $[n]$ of rank $\boldsymbol{r} = (r_1, \dots, r_k)$ respectively. The following statements are equivalent:*

(a) $\boldsymbol{\mu}$ *is a point in* $\mathrm{FlDr}(\boldsymbol{r}; n) = \bigcap_{1 \le i \le k} V(\mathscr{P}^{\mathrm{trop}}_{r_i;n}) \cap \bigcap_{1 \le i < j \le k} V(\mathscr{P}^{\mathrm{trop}}_{r_i,r_j;n})$,

(b) $\boldsymbol{\mu}$ *is a valuated flag matroid,*

(c) $\overline{\mathrm{trop}}(\mu_1) \subseteq \cdots \subseteq \overline{\mathrm{trop}}(\mu_k)$.

Example 3.2.10. In this example we describe the tropicalization of the flag variety $\mathrm{Fl}(1,2;4)$. By definition, $\mathrm{Fl}(1,2;4) = V(\mathscr{P}_{2;4} \cup \mathscr{P}_{1,2;4})$, since $\mathscr{P}_{1;4}$ contains just the zero polynomial. We write down the tropicalizations of the equations defining the ideal:

$$\mathscr{P}^{\mathrm{trop}}_{2;4} = \{p_{1,4}p_{2,3} \oplus p_{1,3}p_{2,4} \oplus p_{1,2}p_{3,4}\},$$

$$\mathscr{P}^{\mathrm{trop}}_{1,2;4} = \left\{ \begin{array}{l} p_1p_{2,3} \oplus p_2p_{1,3} \oplus p_3p_{1,2}, \\ p_4p_{1,2} \oplus p_2p_{1,4} \oplus p_1p_{2,4}, \\ p_4p_{1,3} \oplus p_1p_{3,4} \oplus p_3p_{1,4}, \\ p_4p_{2,3} \oplus p_2p_{3,4} \oplus p_3p_{2,4} \end{array} \right\}.$$

The tropicalization $\mathrm{trop}\big(\mathrm{Fl}(1,2;4)\big)$ can be computed in Macaulay2 [58] and is a 7-dimensional simplicial fan in $\mathbb{R}^{10}$ with a lineality space of dimension 5 and f-vector $(1,10,15)$ after taking the quotient by the lineality space. The tropical variety modulo lineality space is a Petersen graph (see Figure 3.2).

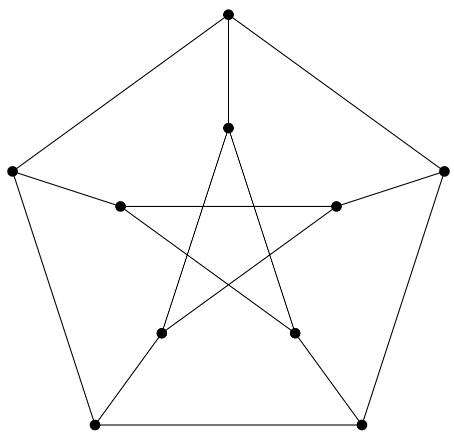


Figure 3.2: Petersen graph

A point in a top-dimensional cone (i.e., a cone over an edge of the Petersen graph) parametrizes a generic tropical line in 4-space with two vertices containing a point. The first vertex can be freely chosen. Then, after accounting for symmetry, there are three choices for the direction of the two outgoing ends of the vertex. This fixes the directions of the edge and the remaining legs by balancing. Finally, the length of the bounded edge can be freely chosen, as can the position of the point on the line. The specific edge of the Petersen graph on which a point lies provides the information about the direction of the outgoing legs, and indicates on which leg or edge the point lies. In total, this generates a cone of dimension seven. $\diamondsuit$

3.3 Linear Degenerate Flag Dressian

In this section, we define the linear degenerate flag Dressian and prove the equivalences of our main results, Theorem A and Theorem B.

3.3.1 Linear degenerate Plücker relations

We start by defining the linear degenerate Plücker relations.

Definition 3.3.1 (Linear degenerate Plücker relations)**.** Let $r \leq s \leq n$ be nonnegative integers and let $S \subseteq [n]$. The *linear degenerate Plücker relations* are the polynomials in the variables $\{p_I : I \in \binom{[n]}{r}\} \cup \{p_J : J \in \binom{[n]}{s}\}$ with coefficients in K:

$$\mathscr{P}_{r,s;S;n} = \left\{ \sum_{j \in J \setminus (I \cup S)} \operatorname{sgn}(j; I, J) p_{I \cup j} p_{J \setminus j} : I \in \binom{[n]}{r-1}, J \in \binom{[n]}{s+1} \right\}$$

where $\operatorname{sgn}(j; I, J) = (-1)^{\#\{j' \in J : j < j'\} + \#\{i \in I : i > j\}}$. We denote their tropicalizations by $\mathscr{P}^{\text{trop}}_{r,s;S;n}$.

Note that with this notation we have $\mathscr{P}_{r,s;\emptyset;n} = \mathscr{P}_{r,s;n}$.

Linear degenerate Plücker relations appear in [24, Section 5.1], arising as initial degenerations of the Plücker relations. The form in which we are expressing them here can be deduced from the relations given in [82]. The linear degenerate Plücker relations parametrize linear degenerate flags of linear spaces. For the sake of completeness, we give a proof similar to the original proof of the classical Plücker relations following [10, Theorem 1.8].

First, we introduce some more notation. For a subset of indices $S \subseteq [n]$, we define the linear map $\operatorname{pr}_S : K^n \to K^n$ by $\operatorname{pr}_S(e_i) = 0$ if $i \in S$ and $\operatorname{pr}_S(e_i) = e_i$ otherwise.

Proposition 3.3.2. *Let U and V be vector subspaces of K^n of dimension $r \leq s$ respectively, and let $S \subseteq [n]$. We have $\operatorname{pr}_S(U) \subseteq V$ if and only if the Plücker coordinates of U and V satisfy the linear degenerate Plücker relations $\mathscr{P}_{r,s;S;n}$.*

Proof. Suppose that $\operatorname{pr}_S(U) \subseteq V$. Let $A \in K^{r,n}$ be a matrix whose rows are a basis of U, and let $A' \in K^{r,n}$ be the matrix obtained from A by substituting the columns indexed by S with columns of zeros. Note that the rows of A' are a set of generators for $\operatorname{pr}_S(U)$. Let $B \in K^{s,n}$ be a matrix whose rows are a basis of V, obtained by extending a basis of $\operatorname{pr}_S(U)$ consisting of rows of A'. Fix $I = \{i_1 < \cdots < i_{r-1}\} \in \binom{[n]}{r-1}$ and $J = \{j_1 < \cdots < j_{s+1}\} \in \binom{[n]}{s+1}$. The column vectors $B_{j_1}, \ldots, B_{j_{s+1}}$ are linearly dependent, and they satisfy the unique (up to scalar) dependency relation

$$\sum_{k=1}^{s+1}(-1)^k \det(B_{j_1},\ldots,B_{j_{k-1}},B_{j_{k+1}},\ldots,B_{j_{s+1}})\cdot B_{j_k}=0.$$

In particular, from the construction of A' and B we also obtain

$$\sum_{k=1}^{s+1}(-1)^k \det(B_{j_1},\ldots,B_{j_{k-1}},B_{j_{k+1}},\ldots,B_{j_{s+1}})\cdot A'_{j_k}=0.$$

Substituting the above expression of the (r-dimensional) zero vector in the equality $\det(0,A_{i_1},\ldots,A_{i_{r-1}})=0$ we obtain

$$\det\left(\sum_{k=1}^{r+1}(-1)^k \det(B_{j_1},\ldots,B_{j_{k-1}},B_{j_{k+1}},\ldots,B_{j_{s+1}})\cdot A'_{j_k},A_{i_1},\ldots,A_{i_{r-1}}\right)=$$
$$\sum_{k=1}^{r+1}(-1)^k \det(B_{j_1},\ldots,B_{j_{k-1}},B_{j_{k+1}},\ldots,B_{j_{s+1}})\cdot \det(A'_{j_k},A_{i_1},\ldots,A_{i_{r-1}})=0.$$

Now by construction we have

$$A'_{j_k}=\begin{cases}A_{j_k} & \text{if } j_k\notin S,\\ 0 & \text{if } j_k\in S.\end{cases}$$

Thus, it is possible to check that the above relations are the desired linear degenerate Plücker relations, up to a possible change of sign that depends on r and s.

Conversely, suppose that the Plücker coordinates of U and V satisfy the linear degenerate Plücker relations. Let A and A' be as above, and let $B\in K^{s,n}$ be a matrix whose rows are a basis of V. We need to show that the rows of A' are spanned by the rows of B. Let $I=\{i_1<\cdots<i_{r-1}\}\in\binom{[n]}{r-1}$ and $J=\{j_1<\cdots<j_{s+1}\}\in\binom{[n]}{s+1}$. Proceeding similarly as above, from the incidence Plücker relations we can write

$$\det\left(\sum_{k=1}^{r+1}(-1)^k \det(B_{j_1},\ldots,B_{j_{k-1}},B_{j_{k+1}},\ldots,B_{j_{s+1}})\cdot A'_{j_k},A_{i_1},\ldots,A_{i_{r-1}}\right)=0. \tag{3.1}$$

Since A has maximal rank, we can choose a subset $I'\in\binom{[n]}{r}$ such that the columns of A indexed by I' form a basis. By choosing all possible cardinality $r-1$ subsets $I\subseteq I'$ in (3.1), we have that the first vector in the argument of the determinant in (3.1) is in the span of the spaces generated by the vectors indexed by all such sets I. This is possible only for the zero vector. Therefore, we obtain:

$$\sum_{k=1}^{r+1}(-1)^k \det(B_{j_1},\dots,B_{j_{k-1}},B_{j_{k+1}},\dots,B_{j_{s+1}})\cdot A'_{j_k}=0.$$

Let C be the matrix consisting of the rows of B plus an additional row of A'. By using Laplace expansion on C with respect to the row of A', the above dependencies tell us that the rank of C is equal to the rank of B, i.e., that the row of A' in C is a linear combination of the rows of B. □

Let $r_1 \le \cdots \le r_k \le n$ be nonnegative integers, let $S_1,\dots,S_k \subseteq [n]$, and denote by $\mathbf{r}=(r_1,\dots,r_k)$, $\mathbf{S}=(S_1,\dots,S_{k-1})$, and $S_{ij}=S_i\cup S_{i+1}\cup\cdots\cup S_{j-1}$ for $1\le i<j\le k$.

Definition 3.3.3 (Linear degenerate flag variety)**.** The *linear degenerate flag variety* of rank $\mathbf{r}$ and degeneration type $\mathbf{S}$ is the following subvariety of $\mathbb{P}^{\binom{n}{r_1}-1}\times\cdots\times\mathbb{P}^{\binom{n}{r_k}-1}$

$$\mathrm{LFl}(\mathbf{r},\mathbf{S};n)=V\big(\{\mathscr{P}_{r_i;n}\}_{1\le i\le k}\cup\{\mathscr{P}_{r_i,r_j;S_{ij};n}\}_{1\le i<j\le k}\big).$$

We call its tropicalization the *linear degenerate tropical flag variety.*

A consequence of Proposition 3.3.2 is $(a)\Leftrightarrow(b)$ of Theorem B. Before proving it, we introduce some more notation. First, we recall the definition of *deletion* for valuated matroids.

Proposition-Definition 3.3.4 ([35, Proposition 1.2])**.** *Let μ be a rank-r matroid on $[n]$ with underlying matroid M, and let $S\subseteq[n]$. Let k be the rank of the deletion $M\setminus S$. Choose $I\in\binom{S}{r-k}$ such that $([n]\setminus S)\cup I$ has rank r. Then, the map $\mu\setminus S:\binom{[n]\setminus S}{k}\to\mathbb{T}$ defined by $(\mu\setminus S)(B)=\mu(B\cup I)$ is a valuated matroid, with underlying matroid $M\setminus S$. Further, $\mu\setminus S$ is compatible with equivalence, and different choices of I give rise to equivalent valuated matroids. The matroid $\mu\setminus S$ is called the* deletion *of μ by $S\subseteq[n]$.*

Let μ be a valuated matroid on $[n]$. We denote by μ_S the valuated matroid on $[n]$ obtained from μ in the following way. For every $B\in\binom{[n]}{k}$, where k is the rank of $\mu\setminus S$, we set

$$\mu_S(B)=\begin{cases}(\mu\setminus S)(B) & \text{if } B\cap S=\emptyset,\\ \infty & \text{otherwise.}\end{cases}\tag{3.2}$$

The matroid μ_S can be alternatively regarded as a direct sum of valuated matroids:

$$\mu_S=(\mu\setminus S)\oplus U_{0,|S|},$$

where we add the deleted elements of the ground set as loops. For a definition of direct sum of valuated matroids, see [65, Definition 2.6].

Recall that we can view (an equivalence class of) a valuated matroid $\mu : \binom{[n]}{r} \to \mathbb{T}$ as a point $\mu \in \mathbb{P}(\mathbb{T}^{\binom{n}{r}})$, and similarly a pair of valuated matroids (μ, ν) of rank r and s respectively, can be viewed as a point $\mu \times \nu \in \mathbb{P}(\mathbb{T}^{\binom{n}{r}}) \times \mathbb{P}(\mathbb{T}^{\binom{n}{s}})$. We are now ready to prove $(a) \Leftrightarrow (b)$ of Theorem B.

Corollary 3.3.5. *Let μ and ν be two realizable valuated matroids of rank r and s, on the common ground set $[n]$, and let $S \subseteq [n]$. Then, $\mu \times \nu \in \overline{\mathrm{trop}}(\mathrm{LFl}(r, s, S; n))$ if and only if $\mu_S \twoheadleftarrow \nu$ is a realizable quotient of valuated matroids.*

Proof. If $\mu \times \nu \in \overline{\mathrm{trop}}(\mathrm{LFl}(r, s, S; n))$ then, from the fundamental theorem of tropical geometry [85], there exist realizations U of μ and V of ν such that the Plücker coordinates of U and V are a point of $\mathrm{LFl}(r, s, S; n)$. From Proposition 3.3.2 this implies that $\mathrm{pr}_S(U) \subseteq V$. Now note that, by definition, the valuated matroid of $\mathrm{pr}_S(U)$ is μ_S, therefore, from Theorem 3.2.9 we have that $\mathrm{pr}_S(U) \subseteq V$ implies $\mu_S \twoheadleftarrow \nu$ and the last quotient is realizable.

Conversely, assume that $\mu_S \twoheadleftarrow \nu$ is a realizable quotient. By definition, μ_S is realizable. Since μ and ν are both realizable, this means that there exist realizations U of μ and V of ν are such that $\mathrm{pr}_S(U) \subseteq V$. From Proposition 3.3.2 this implies that the Plücker coordinates of U and V satisfy the linear degenerate Plücker relations. Therefore the Plücker coordinates of their valuated matroids μ and ν are tropicalization of the Plücker coordinates, that is $\mu \times \nu \in \overline{\mathrm{trop}}(\mathrm{LFl}(r, s, S; n))$. □

Example 3.3.6. In this example we describe the tropicalization of the linear degenerate flag variety $\mathrm{LFl}((1, 2), \{1\}; 4)$. By definition $\mathrm{LFl}((1, 2), \{1\}; 4) = V(\mathscr{P}_{2;4} \cup \mathscr{P}_{1,2;\{1\};4})$, since $\mathscr{P}_{1;4}$ contains just the zero polynomial. We write down the tropicalizations of the equations defining the ideal:

$$\mathscr{P}^{\mathrm{trop}}_{2;4} = \{p_{1,4}p_{2,3} \oplus p_{1,3}p_{2,4} \oplus p_{1,2}p_{3,4}\},$$

$$\mathscr{P}^{\mathrm{trop}}_{1,2;\{1\};4} = \left\{ \begin{array}{l} p_3p_{1,2} \oplus p_2p_{1,3}, \\ p_4p_{1,2} \oplus p_2p_{1,4}, \\ p_4p_{1,3} \oplus p_3p_{1,4}, \\ p_4p_{2,3} \oplus p_3p_{2,4} \oplus p_2p_{3,4}. \end{array} \right\}$$

Note that the polynomials in $\mathscr{P}^{\mathrm{trop}}_{1,2;\{1\};4}$ are obtained from those in $\mathscr{P}^{\mathrm{trop}}_{1,2;4}$ by deleting all monomials containing p_1 (compare with Example 3.2.10). The tropicalization $\mathrm{trop}\big(\mathrm{LFl}((1, 2), \{1\}; 4)\big)$ can be computed in Macaulay2 [58] and is a 7-dimensional simplicial fan in $\mathbb{R}^{10}$. Its lineality space has dimension 6 and the quotient of the variety by the lineality space has f-vector $(1, 3)$.

A point p in $\operatorname{trop}\big(\mathrm{LFl}((1,2),\{1\};4)\big)$ corresponds to a tropical line L_p in 4-space with a point v_p contained after (tropical) projection in direction x_1. This containment can only be satisfied if L_p has a ray in direction x_1. The three maximal cones correspond to the (after symmetry) three possible choices for direction vectors turning L_p into a balanced polyhedral complex. The dimensions of the cones can be derived in a similar argument as in Example 3.2.10. ◊

Definition 3.3.7 (Linear degenerate flag Dressian)**.** The *linear degenerate flag Dressian* of rank $\mathbf{r}$ and degeneration type $\mathbf{S}$ is the tropical prevariety

$$\mathrm{LFlDr}(\mathbf{r},\mathbf{S};n) \subseteq \mathbb{P}(\mathbb{T}^{\binom{n}{r_1}}) \times \cdots \times \mathbb{P}(\mathbb{T}^{\binom{n}{r_k}})$$

given by the intersection of the tropical hypersurfaces of the tropical polynomials in $\{\mathscr{P}^{\mathrm{trop}}_{r_i;n}\}_{1\leq i\leq k} \cup \{\mathscr{P}^{\mathrm{trop}}_{r_i,r_j;S_{ij};n}\}_{1\leq i<j\leq k}$.

Now, for $S \subseteq [n]$, define the projection map $\mathrm{pr}^{\mathrm{trop}}_S : \mathbb{T}^n \to \mathbb{T}^n$ by

$$\Big(\mathrm{pr}^{\mathrm{trop}}_S(x_1,\ldots,x_n)\Big)_i = \begin{cases} x_i & \text{if } i \notin S, \\ \infty & \text{if } i \in S. \end{cases}$$

The projection $\mathrm{pr}^{\mathrm{trop}}_S$ does not give us a well-defined map on the tropical projective space $\mathbb{P}(\mathbb{T}^n)$. Denote by $\varphi : \mathbb{T}^n \setminus \{(\infty,\ldots,\infty)\} \to \mathbb{P}(\mathbb{T}^n)$ the natural quotient map. By abuse of notation, for a subset $X \subseteq \mathbb{P}(\mathbb{T}^n)$ we set

$$\mathrm{pr}^{\mathrm{trop}}_S(X) = \varphi\Big(\mathrm{pr}^{\mathrm{trop}}_S\big(\varphi^{-1}(X)\big) \setminus \{(\infty,\ldots,\infty)\}\Big).$$

Definition 3.3.8 (Linear degenerate tropical flag)**.** A *linear degenerate tropical flag* of degeneration type $\mathbf{S} = (S_1,\ldots,S_{k-1})$, with $S_i \subseteq [n]$, is a sequence of tropical linear spaces $(\overline{\operatorname{trop}}(\mu_1),\ldots,\overline{\operatorname{trop}}(\mu_k))$ on $\mathbb{P}(\mathbb{T}^n)$ such that for all $i \in \{1,\ldots,k-1\}$ we have $\mathrm{pr}^{\mathrm{trop}}_{S_i}(\overline{\operatorname{trop}}(\mu_i)) \subseteq \overline{\operatorname{trop}}(\mu_{i+1})$.

A picture of a linear degenerate tropical flag can be found in Figure 3.1(b).

To show that points in the linear degenerate flag Dressian parametrize linear degenerate tropical flags, and thus $(a) \Leftrightarrow (c)$ in Theorem A, we give an equivalent definition of tropical linear spaces in terms of *cocircuits*.

Definition 3.3.9. The *dual* of a valuated matroid μ is the valuated matroid μ^* defined by $\mu^*(I) = \mu([n]\setminus I)$ for all $I \in \binom{[n]}{d}$. The *valuated cocircuits* of μ are the valuated circuits of μ^*. For each $I \in \binom{[n]}{r-1}$ define $C^*_\mu(I) \in \mathbb{T}^n$ by

$$C^*_\mu(I)_i = \begin{cases} \mu(I \cup i) & i \notin I, \\ \infty & i \in I. \end{cases}$$

The set of *valuated cocircuits* $\mathcal{C}^*(\mu)$ is the image in $\mathbb{P}(\mathbb{T}^n)$ of the following set:

$$\left\{ C^*_\mu(I) : I \in \binom{[n]}{r-1} \right\} \setminus \{(\infty, \dots, \infty)\}.$$

Let μ be a valuated matroid on $[n]$. Its *tropical linear space* can be equivalently defined as the span of its cocircuits (see, for instance, [19, Theorem B]):

$$\overline{\operatorname{trop}}(\mu) = \left\{ \bigoplus_{C \in \mathcal{C}^*(\mu)} \lambda_C \odot C : \lambda_C \in \mathbb{T}, \lambda_C \neq \infty \right\}. \tag{3.3}$$

Proposition 3.3.10. *Let μ and ν be two valuated matroids on a common ground set $[n]$, of rank r and s respectively, and let $S \subseteq [n]$. The following statements are equivalent:*

(1) $\mu \times \nu \in \mathrm{LFlDr}(r, s, S; n)$,

(2) $\mathrm{pr}^{\mathrm{trop}}_S(\overline{\operatorname{trop}}(\mu)) \subseteq \overline{\operatorname{trop}}(\nu)$.

Proof. Each μ and ν satisfy its respective tropical Grassmann-Plücker relations if and only if μ and ν are valuated matroids respectively.

Now $\mu \times \nu \in \mathrm{LFlDr}(r, s, S; n)$ if and only if for every $I \in \binom{[n]}{r-1}$, $J \in \binom{[n]}{s+1}$ the minimum in

$$\bigoplus_{j \in J \setminus (I \cup S)} p_{I \cup j} p_{J \setminus j}.$$

is achieved at least twice. Since, from Definition 3.3.9, we have

$$\left(\mathrm{pr}^{\mathrm{trop}}_S \big(C^*_\mu(I) \big) \right)_j = \begin{cases} \mu(I \cup j) & \text{if } j \notin I \cup S \\ \infty & \text{otherwise,} \end{cases}$$

the above statement is equivalent to requiring that, for every $I \in \binom{[n]}{r-1}$ and $J \in \binom{[n]}{s+1}$, the minimum in

$$\begin{aligned} \bigoplus_{j \in J \setminus (I \cup S)} \mu(I \cup j) \odot \nu(J \setminus j) &= \bigoplus_{j \in J \setminus (I \cup S)} C^*_\mu(I)_j \odot C_\nu(J)_j \\ &= \bigoplus_{j \in [n]} \left(\mathrm{pr}^{\mathrm{trop}}_S \big(C^*_\mu(I) \big) \right)_j \odot C_\nu(J)_j \end{aligned}$$

is achieved at least twice. This holds true if and only if for every valuated cocircuit $C^*_\mu(I)$ of μ and every valuated circuit $C_\nu(J)$ of ν, we have

$$\mathrm{pr}_S(C^*_\mu(I)) \in V\left(\bigoplus_{j\in[n]} C_\nu(J)_j \odot x_i\right).$$

The above statement is equivalent to $\mathrm{pr}_S(C^*_\mu(I)) \in \overline{\mathrm{trop}}(\nu)$ for every valuated cocircuit $C^*_\mu(I)$. By (3.3) and the fact that tropical linear spaces are tropically convex [32], this is equivalent to $\mathrm{pr}^{\mathrm{trop}}_S(\overline{\mathrm{trop}}(\mu)) \subseteq \overline{\mathrm{trop}}(\nu)$. □

This proves $(a) \Leftrightarrow (c)$ in Theorem A.

Remark 3.3.11. One could define the linear degenerate flag Dressian with just the "consecutive" incidence Plücker relations $\mathscr{P}^{\mathrm{trop}}_{r_i,r_{i+1};S_{i,i+1};n}$, instead of taking all the relations $\mathscr{P}^{\mathrm{trop}}_{r_i,r_j;S_{i,j};n}$ for $1 \le i < j \le n$. One of the consequences of the previous proposition is that these two a priori different ways of defining the linear degenerate flag Dressian give rise to the same tropical prevariety.

Remark 3.3.12. As observed in Remark 3.2.8 for the flag Dressian and the flag variety, the tropicalization of the linear degenerate flag variety and the linear degenerate flag variety are, in general, different. In general, the linear degenerate tropical flag variety and the linear degenerate flag Dressian can be different, as the tropical flag variety and the flag Dressian differ, see [19, Example 5.2.4]. Further, even the linear degenerate flag varieties with highest degeneration (i.e. $S_i = [n]$ for all i in Definition 3.3.7) are expected to be different. They are products of tropicalized Grassmannians and products of Dressians respectively, and Dressians and tropical Grassmannians are different for large enough k and n.

3.3.2 Linear degenerate valuated flag matroids

In this section, we prove the equivalence $(b) \Leftrightarrow (c)$ of Theorem A and Theorem B, that is, (realizable) linear degenerate valuated flag matroids correspond to (realizable) linear degenerate tropical flags. We begin by defining linear degenerate valuated flag matroids. Again, for a valuated matroid μ on $[n]$ and a subset $S \subseteq [n]$ we are going to use our notation μ_S defined in (3.2). More explicitly, the valuated circuits of the deletion $\mu \setminus S$ are

$$\mathcal{C}(\mu \setminus S) = \left\{ C_{|[n]\setminus S} : C \in \mathcal{C}(\mu), \mathrm{supp}(C) \subseteq [n] \setminus S \right\}. \tag{3.4}$$

This is a direct consequence of [90, Theorem 3.1]. For the formulation used here, see [19, Theorem 3.1.6].

Definition 3.3.13 (Linear degenerate valuated flag matroid)**.** A sequence $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_k)$ of valuated matroids on $[n]$ is a *linear degenerate valuated flag matroid* if for all $i \in \{1, \ldots, k-1\}$, there exists $S_i \subseteq [n]$ such that $(\mu_i)_{S_i} \twoheadleftarrow \mu_{i+1}$. In addition, $\boldsymbol{\mu}$ is *realizable* if the quotient $(\mu_i)_{S_i} \twoheadleftarrow \mu_{i+1}$ is realizable for all $i \in \{1, \ldots, k-1\}$ using the same realization L_i for the quotients $(\mu_{i-1})_{S_{i-1}} \twoheadleftarrow \mu_i$ and $(\mu_i)_{S_i} \twoheadleftarrow \mu_{i+1}$.

Proposition 3.3.14. *Let μ be a valuated matroid on the ground set $[n]$ and let $S \subset [n]$. Then*

$$\mathrm{pr}^{\mathrm{trop}}_{\{S\}}(\overline{\mathrm{trop}}(\mu)) = \overline{\mathrm{trop}}\,(\mu \setminus S) \times \{\infty\}^{\{S\}} = \overline{\mathrm{trop}}(\mu_S).$$

Proof. The second equality follows from the definition of μ_S. Now we prove the first equality. First, we note that we can restrict to the case $S = \{s\}$ and obtain the result for arbitrary S by inductively re-applying the one-element case.

Let $v \in \overline{\mathrm{trop}}(\mu)$. Then the minimum in $\{C_i + v_i\}_{i\in[n]}$ is achieved at least twice for every $C \in \mathcal{C}(\mu)$. In particular, the minimum in $\{C_i + v_i\}_{i\in[n]\setminus s}$ is achieved at least twice for every $C \in \mathcal{C}(\mu)$ where $\mathrm{supp}(C) \subseteq [n] \setminus s$. From (3.4), $\mathrm{pr}^{\mathrm{trop}}_{\{s\}}(v) \in \overline{\mathrm{trop}}(\mu \setminus s) \times \{\infty\}^{\{s\}}$. This proves the first inclusion.

For the reverse inclusion, let $v \in \overline{\mathrm{trop}}(\mu \setminus s) \times \{\infty\}^{\{s\}}$. Then, the minimum in $\{C_i + v_i\}_{i\in[n]\setminus s}$ is achieved at least twice for every $C \in \mathcal{C}(\mu \setminus s)$. From (3.4) this means it is achieved at least twice for every $C \in \mathcal{C}(\mu)$ with $\mathrm{supp}(C) \subseteq [n] \setminus s$. Now we want to find some $t \in \mathbb{T}$ such that the vector $\tilde{v} = (v_1, \ldots, v_{s-1}, t, v_{s+1}, \ldots, v_n) \in \mathbb{P}(\mathbb{T}^n)$, is in $\overline{\mathrm{trop}}(\mu)$. Then $v = \mathrm{pr}^{\mathrm{trop}}_{\{s\}}(\tilde{v}) \in \mathrm{pr}^{\mathrm{trop}}_{\{s\}}(\overline{\mathrm{trop}}(\mu))$.

If for every $C \in \mathcal{C}(\mu)$ the minimum in $\{C_i + v_i\}_{i\in[n]\setminus s}$ is achieved at least twice, we can set $t = \infty$ and we are done. Therefore, we assume that there exists a circuit $C \in \mathcal{C}(\mu)$ such that the minimum in $\{C_i + v_i\}_{i\in[n]\setminus s}$ is achieved only once. Let $t \in \mathbb{R}$ such that $t + C_s = \min_{i\in[n]\setminus s}\{C_i + v_i\}$. Then, the minimum in $\{C_i + \tilde{v}_i\}_{i\in[n]}$ is achieved twice. We claim that $\tilde{v} \in \overline{\mathrm{trop}}(\mu)$.

We proceed by contradiction. Let $C' \in \mathcal{C}(\mu)$ and assume that the minimum in $\{C'_i + \tilde{v}_i\}_{i\in[n]}$ is achieved only once at the index $j \in [n]$. Up to tropical scalar multiplication we can assume that $C'_s = C_s \neq \infty$. Suppose first that $j \neq s$. By construction, $v_i + C_i \geq t + C_s = t + C'_s > v_j + C'_j$ for every $i \in [n]$, in particular $C_j \neq C'_j$. On the other hand, we have $v_i + C'_i > v_j + C'_j$ for every $i \neq j$, therefore $v_i + \min(C_i, C'_i) > v_j + C'_j$ for every $i \neq j$. From [90, Theorem 3.4] there exists a valuated cycle C'' of μ such that $C''_s = \infty$, $C''_i \geq \min\{C_i, C'_i\}$ for all $i \in [n]$ with equality whenever $C_i \neq C'_i$, in particular $C''_j = C'_j$. But now $\mathrm{supp}(C'') \subseteq [n] \setminus s$, so the minimum in $\{C''_i + v_i\}_{i\in[n]\setminus s}$ has to be achieved at least twice, contradicting $v_i + C''_i \geq v_i + \min(C_i, C'_i) > v_j + C'_j$ for every $i \neq j$.

Now suppose that $j = s$. Let $k \in [n]$ be the index at which the minimum in $\{v_i + C_i\}_{i \in [n] \setminus s}$ is achieved. Then we have $v_k + C_k = t + C_s = t + C'_s < v_i + C'_i$ for every $i \neq s$, in particular $C_k < C'_k$. Now, applying [90, Theorem 3.4] again, we obtain a valuated cycle C'' with the same properties as above, in particular $C''_k = C_k$. Then, $v_k + C''_k = v_k + C_k < v_i + C'_i$ and further $v_k + C''_k = v_k + C_k < v_i + C_i$ for every $i \neq s, k$. This contradicts the fact that the minimum in $\{v_i + C''_i\}_{i \in [n] \setminus s}$ is achieved at least twice. $\square$

Theorem 3.3.15. *Let μ and ν be valuated matroids on a common ground set $[n]$. The following statements are equivalent*

(1) $\mu_S \twoheadleftarrow \nu$,

(2) $\mathrm{pr}_S^{\mathrm{trop}}(\overline{\mathrm{trop}}(\mu)) \subseteq \overline{\mathrm{trop}}(\nu)$.

Proof. From Theorem 3.2.9 $(b) \Leftrightarrow (c)$ and Proposition 3.3.14, we have $\mu_S \twoheadleftarrow \nu$ if and only if $\mathrm{pr}_S^{\mathrm{trop}}(\overline{\mathrm{trop}}(\mu)) = \overline{\mathrm{trop}}(\mu_S) \subseteq \overline{\mathrm{trop}}(\nu)$. $\square$

3.3.3 Morphisms of valuated matroids

In this section, we outline how we can recast the definition of linear degenerate valuated flag matroids in terms of morphisms of valuated matroids (as defined in [19, Remark 4.3.3]) and prove Theorem A (b) ⇔ (d). The advantage of doing so is that this allows for further generalizations. For instance one can define a *quiver Dressian* by using morphisms of valuated matroids in place of linear maps between linear spaces.

Let M (or μ) be a (valuated) matroid on the ground set $[m]$. Let o be an element not in $[m]$. We denote by M_o (or μ_o) the matroid $M \oplus U_{0,1}$ (or $\mu \oplus U_{0,1}$) obtained by adding o as a loop.

In addition, let N be a matroid on the ground set $[n]$. A *morphism* (or *strong map*) of matroids $f : M \to N$ is a map of sets $f : [m] \cup \{o\} \to [n] \cup \{o\}$ such that $f(o) = o$ and the inverse image of a flat in N_o is a flat in M_o. Morphisms of matroids can be characterized in terms of matroid quotients.

Definition 3.3.16. Let $f : [m] \to [n]$ be a map of sets and N a matroid over $[n]$. The *induced* matroid $f^{-1}(N)$ on $[m]$ is defined by $\mathrm{rk}_{f^{-1}(N)}(A) = \mathrm{rk}_N(f(A))$ for every $A \subseteq [m]$.

Lemma 3.3.17 ([38, Lemma 2.4])**.** *The map of sets $f : [m] \cup \{o\} \to [n] \cup \{o\}$ with $f(o) = o$ is a morphism of matroids if and only if $f^{-1}(N_o) \twoheadleftarrow M_o$.*

Now we use the above characterization to extend the definition of morphism to valuated matroids, starting from the notion of quotients of valuated matroids.

Proposition 3.3.18. *Let $f : [m] \to [n]$ be a surjective map of sets and let ν be a valuated matroid on $[n]$ of rank r with underlying matroid N. The function $f^{-1}(\nu) : \binom{[m]}{r} \to \mathbb{T}$ defined by*

$$f^{-1}(\nu)(I) = \begin{cases} \nu(f(I)) & \text{if } I \text{ is a basis of } f^{-1}(N), \\ \infty & \text{otherwise,} \end{cases}$$

is a valuated matroid with underlying matroid $f^{-1}(N)$.

Proof. Since f is surjective, N and $f^{-1}(N)$ have the same rank r. Now if $B \in \binom{[m]}{r}$, then from $\mathrm{rk}_{f^{-1}(N)}(B) = \mathrm{rk}_N(f(B))$ we have that B is a basis of $f^{-1}(N)$ if and only if $f(B)$ is a basis of N. This proves that the map $f^{-1}(\nu)$ is well defined. In particular, if B is a basis of $f^{-1}(N)$, then $|B| = |f(B)|$, so the restriction of f on B is bijective.

Now it suffices to show that for $I, J \in \binom{[m]}{r}$ and $i \in I \setminus J$ there exists $j \in J \setminus I$ such that

$$f^{-1}(\nu)(I) + f^{-1}(\nu)(J) \geq f^{-1}(\nu)(I \cup j \setminus i) + f^{-1}(\nu)(J \cup i \setminus j). \tag{3.5}$$

If I or J is not a basis of $f^{-1}(N)$, then the left hand side of the above inequality is ∞ and we are done. Otherwise, assume that I and J are bases of $f^{-1}(N)$. This means that $f(I)$ and $f(J)$ are bases of ν. Now, if $f(i) \in f(I) \setminus f(J)$, then there exists $j \in J$ such that

$$\nu(f(I)) + \nu(f(J)) \geq \nu(f(I) \cup f(j) \setminus f(i)) + \nu(f(J) \cup f(i) \setminus f(j)). \tag{3.6}$$

If $f(i) \in f(J)$, then there exists $j \in J$ such that $f(i) = f(j)$. Therefore, we have the equality in (3.6). In any case, we get the inequality (3.6), which, from the definition of $f^{-1}(\nu)$, is exactly (3.5). $\square$

Definition 3.3.19. Let $f : [m] \to [n]$ be a map of sets and let ν be a valuated matroid on the ground set $[n]$. The *induced* valuated matroid is defined by $f^{-1}(\nu) = f^{-1}(\nu_{|f([m])})$.

Definition 3.3.20. Let μ and ν be valuated matroids on the ground set $[m]$ and $[n]$ respectively. A map of sets $f : [m] \cup \{o\} \to [n] \cup \{o\}$ with $f(o) = o$ is a *morphisms of valuated matroids*, denoted $f : \mu \to \nu$, if $f^{-1}(\nu_o) \twoheadleftarrow \mu_o$.

In our context, it suffices to restrict to projections. Let μ and ν be two valuated matroids on the common ground set $[n]$, and let $S \subseteq [n]$. Define the projection map $\mathrm{pr}_S : [n] \cup \{o\} \to [n] \cup \{o\}$ by

$$\mathrm{pr}_S(x) = \begin{cases} x & \text{if } x \notin S, \\ o & \text{if } x \in S. \end{cases}$$

Using this definition, we prove the final part of Theorem A, (b) $\Leftrightarrow$ (d).

Proposition 3.3.21. *Let μ and ν be two valuated matroids on the common ground set $[n]$, and let $S \subseteq [n]$. Then, $\mathrm{pr}_S : \nu \to \mu$ is a morphism of valuated matroids if and only if $\mu_S \twoheadleftarrow \nu$.*

Proof. Recall the definition of μ_S from (3.2). By construction, we have $\mathrm{pr}_S^{-1}(\mu_o) = (\mu_o)_S$. Thus $\mathrm{pr}_S : \nu \to \mu$ is a morphism of valuated matroids if and only if $(\mu_o)_S \twoheadleftarrow \nu_o$ if and only if $\mu_S \twoheadleftarrow \nu$. $\square$

Note that any linear map realizing the morphism $\mathrm{pr}_S : \nu \to \mu$, after a change of coordinates, is always a projection. This follows directly from the definition of a realizable morphism of matroids, that can be found in [38, Section 2]. This fact, together with the above proposition, prove $(b) \Leftrightarrow (d)$ of Theorem B.

3.4 The poset of linear degenerate flag varieties

Linear degenerate flag varieties can be arranged in a poset in a natural way as follows. Fix $n, k \in \mathbb{N}$ and a sequence $\mathbf{r} = (r_1, \ldots, r_k)$ of nonnegative integers such that we have $r_1 \leq \cdots \leq r_k \leq n$. Define the set of linear degenerate flag varieties

$$\mathcal{L} = \left\{ \mathrm{LFl}(\mathbf{r}, \mathbf{S}; n) : \mathbf{S} = (S_1, \ldots, S_{k-1}) \text{ with } S_i \subseteq [n] \right\}.$$

We can give an order relation $\preceq$ on $\mathcal{L}$ defined by $\mathrm{LFl}(\mathbf{r}, \mathbf{S}; n) \preceq \mathrm{LFl}(\mathbf{r}, \mathbf{S}'; n)$ if and only if $S_i \subseteq S'_i$ for every $i \in \{1, \ldots, k\}$, where $\mathbf{S}' = (S'_1, \ldots, S'_{k-1})$. Note that $(\mathcal{L}, \preceq)$ is a finite lattice isomorphic to the product of lattices $\prod_{i=1}^{k} 2^{[n]}$, where $2^{[n]}$ is the power set of $[n]$ ordered by set inclusion.

The maximum of $\mathcal{L}$ is the linear degenerate flag variety with $S_i = [n]$ for every i, in other words at each step of the flag we are projecting all the coordinates, so the linear spaces of the flag do not have any relation to each other, and the variety we obtain is just a product of Grassmannians:

$$\mathrm{LFl}(\mathbf{r}, ([n], \ldots, [n]); n) = G(r_1; n) \times \cdots \times G(r_k; n).$$

On the other hand, the minimum of $\mathcal{L}$ is the linear degenerate flag variety with $S_i = \emptyset$ for every i. Here, we are not degenerating the flag variety as at each step the projection is an identity map, thus all linear degenerate flags are flags:

$$\mathrm{LFl}(\mathbf{r}, (\emptyset, \ldots, \emptyset); n) = \mathrm{Fl}(\mathbf{r}; n).$$

Analogously, we can arrange linear degenerate tropical flag varieties, and linear degenerate flag Dressians in lattices isomorphic to $\prod_{i=1}^{k} 2^{[n]}$.

Linear degenerate tropical flag varieties with $n = 4$ Now, we want to take a closer look at the lattice of linear degenerate tropical flag varieties for the case $n = 4$.

We used Macaulay2 [58] to compute the linear degenerate Plücker relations, using code available in the Github repository [16]; and used the package `Tropical.m2` [4] to compute the respective linear degenerate tropical flag varieties. We did some additional computations in `gfan` [68] and Oscar [92].

For the rest of this section, we will consider varieties of *complete flags* in $\mathbb{C}^4$. More precisely, we fix $\mathbf{r} = (1, 2, 3)$, and omit $\mathbf{r}$ in our notation. For instance, we denote $\mathrm{Fl}(4) := \mathrm{Fl}(\mathbf{r}; 4)$ and $\mathrm{LFl}\big((\{1\}, \emptyset); 4\big) := \mathrm{LFl}\big(\mathbf{r}, (\{1\}, \emptyset); 4\big)$. To simplify the notation, we will use $\mathrm{LFl}(S_1, S_2; 4)$ in place of $\mathrm{LFl}((S_1, S_2); 4)$.

Example 3.4.1. The tropicalization $\mathrm{trop}(\mathrm{Fl}(4))$ of the flag variety $\mathrm{Fl}(4)$ is a nine-dimensional simplicial fan in $\mathbb{R}^{14}$ with f-vector $(1, 20, 79, 78)$ after quotienting out by the lineality space, and lineality dimension six. This variety was computed in [18, Theorem 4], and it parametrizes full flags of length 4, that is, a point inside a tropical line inside a tropical plane. A picture of a point of $\mathrm{trop}(\mathrm{Fl}_4)$, i.e. a full flag of length 4, was given in Figure 3.1(a). In addition, as explained in [67, Paragraph 3.3.3], after quotienting out by the lineality space, $\mathrm{trop}(\mathrm{Fl}(4))$ can be seen as a "tropical line bundle" over the Petersen graph, parametrizing tropical flags. We give [67, Figure 9] in Figure 3.3. By [19, Theorem 5.2.1], all tropical flags of length 4 are realizable, i.e. $\mathrm{trop}(\mathrm{Fl}(4)) = \mathrm{FlDr}(4)$. Concretely, a point p in $\mathrm{trop}(\mathrm{Fl}(4))$ can be interpreted in the following way. The edge on the Petersen graph that p lies on indicates which green or blue rays in the subdivision of the tropical plane of Figure 3.1 contain the vertices of the tropical line. For instance, the tropical flag in Figure 3.1(a) corresponds to a point on the edge connecting 12 and 34. There are two different types of rays on this Petersen graph - the rays connecting (ab) to (cd) for some $a, b, c, d \in \{1, 2, 3, 4\}$ and the rays connecting (a) to (ab). The corresponding rays in the tropical plane are arranged differently: the rays (ab) and (cd) in the tropical plane span a one-dimensional space, and the rays (a) to (ab) span a two-dimensional cone. This difference is also reflected in their tropical line bundles, see Figure 3.3. Finally, the location on the line bundle indicates where the point of the tropical flag lies on the line. $\diamondsuit$

Example 3.4.2. Now, we want to consider the linear degenerate tropical flag varieties $\mathrm{trop}(\mathrm{LFl}(\{1\}, \emptyset; 4))$ and $\mathrm{trop}(\mathrm{LFl}(\emptyset, \{1\}; 4))$. The tropical variety $\mathrm{trop}(\mathrm{LFl}(\{1\}, \emptyset; 4))$ parametrizes (realizable) tropical flags consisting of a point whose projection with respect to the first coordinate lies on a tropical line that is contained in a tropical plane.

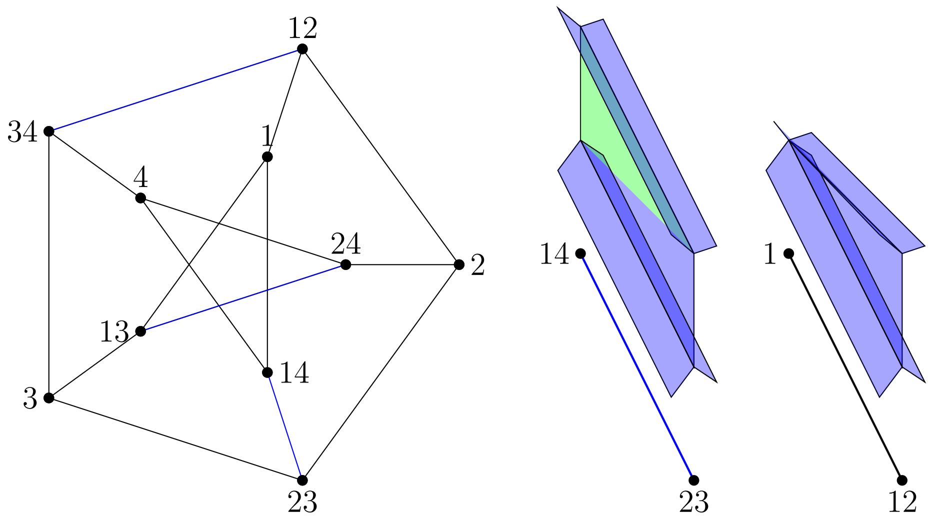


Figure 3.3: The tropical flag variety trop(Fl_4) quotient by its lineality space, interpreted as a "tropical line bundle" over the Petersen graph, see also [67, Figure 9]

This setting was depicted in Figure 3.1 (b). Similarly, trop(LFl($\emptyset, \{1\}; 4$)) parametrizes (realizable) tropical flags consisting of a point contained in a tropical line whose projection with respect to the first coordinate is contained in a tropical plane.

The two different types of linear degenerate flags described above are dual to each other, in fact LFl($\{1\}, \emptyset; 4$) $\simeq$ LFl($\emptyset, \{1\}; 4$). Their tropicalizations trop(LFl($\{1\}, \emptyset; 4$)) and trop(LFl($\emptyset, \{1\}; 4$)) have a similar structure, they both are nine-dimensional simplicial fans in $\mathbb{R}^{14}$ with lineality dimension seven. After quotienting by the lineality space, we get a fan over the Petersen graph which has f-vector $(1, 10, 15)$. The tropical varieties are "usual line bundles" (as opposed to the "tropical line bundles" in Example 3.4.1) over the Petersen graph. In Figure 3.4, we depict this degeneration. Further, we depict the degeneration of the line bundles over the edges on the right of Figure 3.3. $\diamondsuit$

Example 3.4.3. Finally, we consider the linear degenerate tropical flag variety given by trop(LFl($\{1\}, \{1\}; 4$)). It is a nine-dimensional simplicial fan in $\mathbb{R}^{14}$ with f-vector $(1, 3)$ after quotienting out by the lineality space, and lineality dimension eight, i.e. an eight-dimensional "line bundle" over a tropical line. We depict it in Figure 3.5. Only the left line component of the previous Figures 3.3 and 3.4 appears as a top-dimensional cell, only the line bundles over the blue lines survive the degeneration, whereas the line bundles over all black lines in the Petersen graph degenerate into the lineality space. $\diamondsuit$

One possible application of the poset of linear degenerate tropical flag varieties would be to reduce the problem of computing a tropical flag variety to the problem of computing (a product of) tropical Grassmannians. Recall that the tropical flag variety is

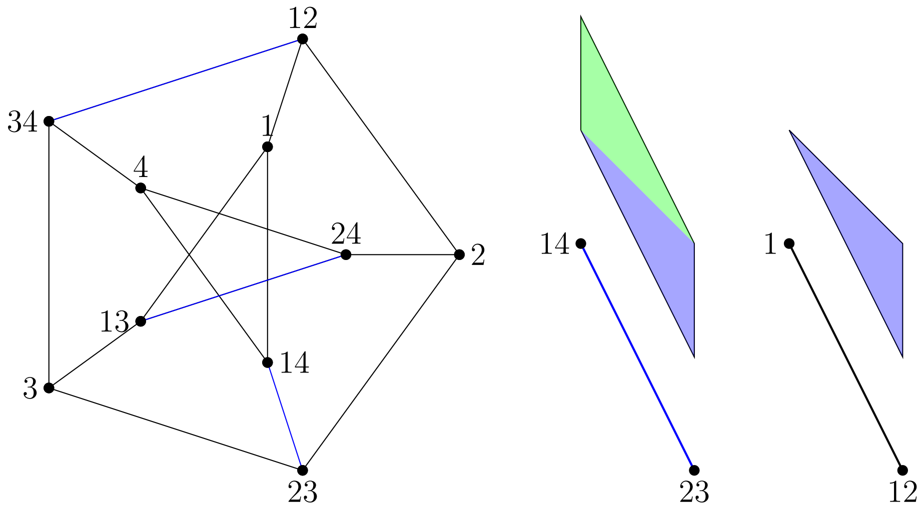


Figure 3.4: The linear degenerate tropical flag variety $\mathrm{trop}(\mathrm{LFl}(\emptyset, \{1\}); 4)$ can be interpreted as a “line bundle" over the Petersen graph.

the minimum of the poset $\mathcal{L}$ of linear degenerate tropical flag varieties, while its maximum is a product of tropical Grassmannians. Therefore, one could try to start from the top of $\mathcal{L}$, and, by descending the poset $\mathcal{L}$ step by step, reconstruct the structure of the tropical flag variety. In order to do that, it would be enough to understand what happens at the *covers* of the poset $\mathcal{L}$, that is, to (fully or partially) reconstruct the structure of $\mathrm{trop}(\mathrm{LFl}(\mathbf{r}, \mathbf{S}; n))$ from another linear degenerate tropical flag variety $\mathrm{trop}(\mathrm{LFl}(\mathbf{r}, \mathbf{S}'; n))$ that covers it, i.e. $\mathbf{S}$ is obtained from $\mathbf{S}'$ by adding one element in one of the sets S_i.

Question 3.4.4. Can we reconstruct the structure of $\mathrm{trop}(\mathrm{LFl}(\mathbf{r}, \mathbf{S}; n))$ from a cover?

The examples we have seen above already provide some insight into what the answer is for full flags with $n = 4$. A common behaviour that we observe is that the lineality space increases in dimension after each linear degeneration. The next result shows that, for an ideal $I \subseteq k[x_0, \ldots, x_n]$, the lineality space of a tropical variety $\mathrm{trop}(V(I))$ contains the homogeneity space of I, which is the linear subspace of vectors $v \in \mathbb{R}^{n+1}$ such that I is homogeneous with respect to the grading $\deg(x_i) = v_i$.

Lemma 3.4.5. *Let $I \subseteq k[x_0, \ldots, x_n]$ be an ideal, where k is a field with the trivial valuation. Let $v = (v_0, \ldots, v_n) \in \mathbb{R}^{n+1}$. If I is homogeneous with respect to the grading $\deg(x_i) = v_i$ then v is in the lineality space of* $\mathrm{trop}(V(I))$.

Proof. If I is homogeneous with respect to the grading $\deg(x_i) = v_i$, then $\mathrm{in}_v(f) = f$ for every $f \in I$. This implies that $\mathrm{in}_{v+w}(f) = \mathrm{in}_w(f)$, as for every monomial m of f, we are adding the same weight to the scalar product of the exponent vector of m and w. In particular $\mathrm{in}_{w+v}(I) = \mathrm{in}_w(I)$ for every $w \in \mathbb{R}^n$. Hence $w \in \mathrm{trop}(V(I))$ if and only if $w + v \in \mathrm{trop}(V(I))$, that is, v is in the lineality space of $\mathrm{trop}(V(I))$. $\square$

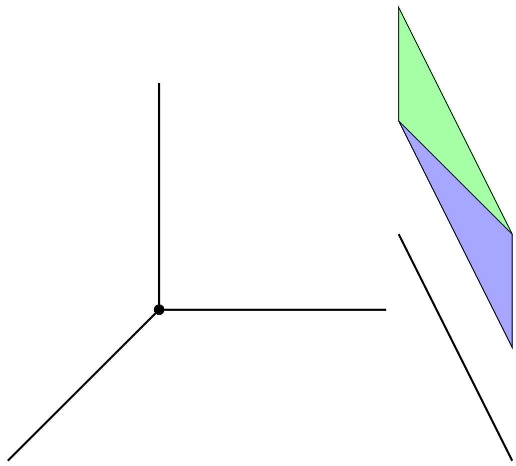


Figure 3.5: The linear degenerate tropical flag variety trop(LFl($\{1\}, \{1\}$); 4) can be interpreted as a “line bundle" over the tropical line.

Corollary 3.4.6. *The homogeneity space of a linear degenerate tropical flag variety is contained in the homogeneity space of any one that covers it.*

Proof. The claim follows from the structure of the linear degenerate Plücker relations. In fact, fixed a grading of the Plücker variables, if the polynomials in $\mathscr{P}_{r,s;S;n}$ are homogeneous with respect to this grading, then this will be also the case for the polynomials in $\mathscr{P}_{r,s;S';n}$ for every $S' \supseteq S$. $\square$

By looking at the examples in the previous section, one might be tempted to conjecture that a cover relation on the poset implies set inclusion on the tropical varieties. In general, this is false, as the following example shows.

Example 3.4.7. In this example, we are going to show that

$$\mathrm{trop}\left(\mathrm{LFl}((1,2),\emptyset;4)\right) \not\subseteq \mathrm{trop}\left(\mathrm{LFl}((1,2),\{1\};4)\right).$$

We already described the above tropical varieties in Example 3.2.10 and Example 3.3.6. Now, assume that our base field K is the *field of Laurent series* $\mathbb{K}((t))$, that is the quotient field of the DVR $\mathbb{K}[[t]]$ of formal power series with coefficients in a field $\mathbb{K}$ in the variable t. Then, K has valuation $v : K \to \mathbb{T}$ where $v(f(t))$ is the minimum of the exponents appearing in f.

Now let $a, b \in \mathbb{Q}$ with $b > a > 0$, and consider the two matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ t^a & 0 & t^b & 1 \end{pmatrix}.$$

Let $L_1, L_2 \subseteq \mathbb{K}^4$ be the two linear spaces generated by the rows of the matrices A_1 and A_2 respectively. By construction, $L_1 \subseteq L_2$, but $\mathrm{pr}_{\{1\}}(L_1) \nsubseteq L_2$. We can see this through the Plücker equations computed in Example 3.3.6. The valuations of the Plücker coordinates of L_1 and L_2 are:

$$(p_1, p_2, p_3, p_4) = (0, 0, 0, 0),$$
$$(p_{1,2}, p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4}) = (a, a, 0, b, 0, 0).$$

In particular, the minimum in all the tropical polynomials in $\mathscr{P}^{\mathrm{trop}}_{2;4}$ and $\mathscr{P}^{\mathrm{trop}}_{1,2;4}$ are achieved at least twice, while the minimum in, for instance, the second tropical polynomial of $\mathscr{P}^{\mathrm{trop}}_{1,2;\{1\};4}$ listed in Example 3.3.6, $p_4 p_{1,2} \oplus p_2 p_{1,4}$ is not achieved twice:

$$p_4 p_{1,2} = 0 \odot a = a > 0 = 0 \odot 0 = p_2 p_{1,4}. \qquad \diamondsuit$$

While we do not obtain containment on tropical flag varieties or Dressians in the poset of linear degenerations, from the definition of the linear degenerate Plücker relations, we obtain the following containment on some boundary components.

Corollary 3.4.8. *Let* $\mathrm{LFlDr}(r, r', S \cup \{s\}, n) \prec \mathrm{LFlDr}(r, r', S, n)$ *be a cover in the poset of linear degenerate flag Dressians. Set*

$$\mathscr{B} = \left\{ (p_I) \in \mathbb{T}^{\binom{n}{r}} \times \mathbb{T}^{\binom{n}{r'}} : p_I = \infty \textit{ for every } I \in \binom{[n]}{r} \textit{ such that } s \in I \right\}.$$

Then we have

$$\mathrm{LFlDr}(r, r', S, n) \cap \mathscr{B} \subseteq \mathrm{LFlDr}(r, r', S \cup \{s\}, n).$$

Another interesting application of the poset of linear degenerate flag varieties concerns *relative realizability.* We say that two realizable tropical linear spaces $T_1 \subseteq T_2$ are *relatively realizable* if there exist realizations L_1 of T_1 and L_2 of T_2 such that $L_1 \subseteq L_2$. Let $\mathcal{L}$ be the poset of linear degenerate tropical flag varieties with flags of length 2 in $\mathbb{P}(\mathbb{T}^n)$ and rank vector (r, s). Then, accurately describing the cover relations of $\mathcal{L}$ might provide us a way to solve the relative realizability problem. In fact, the maximal element of $\mathcal{L}$ is $\mathrm{trop}(G(r;n)) \times \mathrm{trop}(G(s;n))$ in which we impose no conditions on either containment or relative realizability, whereas the minimal element $\mathrm{trop}(\mathrm{Fl}(r, s; n))$ of $\mathcal{L}$ does. Thus, if we could explicitly reconstruct $\mathrm{trop}(\mathrm{Fl}(r, s; n))$ from $\mathrm{trop}(G(r;n)) \times \mathrm{trop}(G(s;n))$, we would have an explicit solution to the relative realizability problem by tracking elements in the cover relations.

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