Set of independencies and Tutte polynomial of matroids over a domain

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Matroids over a ring

Definition (Fink, Moci)

A **matroid over a ring** R on the ground set E is a function $\mathcal{M}$ that assign to each subset A of E a finitely generated R-module $\mathcal{M}(A)$ in such a way that for every $b, c \in E \setminus A$, there exists $x, y \in \mathcal{M}(A)$ such that

$$\mathcal{M}(A \cup \{b\}) \simeq \mathcal{M}(A)/(x)$$
$$\mathcal{M}(A \cup \{c\}) \simeq \mathcal{M}(A)/(y)$$
$$\mathcal{M}(A \cup \{b, c\}) \simeq \mathcal{M}(A)/(x, y)$$

(note that the choice of x and y depends on both b and c).

The property required in the above definition is summarized by the following diagram:

$$\begin{array}{ccc} \mathcal{M}(A) & \xrightarrow{/(x)} & \mathcal{M}(A \cup \{a\}) \\ \downarrow{\scriptstyle /(y)} & & \downarrow{\scriptstyle /(\overline{y})} \\ \mathcal{M}(A \cup \{b\}) & \xrightarrow{/(\overline{x})} & \mathcal{M}(A \cup \{a, b\}) \end{array}$$

Example

Let $R = \mathbb{Z}[i]$, $E = \{1, 2\}$ and consider the matrix

$$(v_1, v_2) = \begin{bmatrix} 1 & 1+i \\ 1+i & 0 \end{bmatrix} \in R^{2,2},$$

Now let $\psi : E \to R^2$ defined by $\psi(i) = v_i$, and define $\mathcal{M} : 2^E \to R$-mod by

$$\mathcal{M}(A) = \frac{R^2}{\langle \psi(i) : i \in A \rangle}, \quad \text{for every } A \subseteq E.$$

Thus $\mathcal{M}$ is a *realizable* R-matroid and ψ is a realization.

$$\begin{array}{ccc} \mathcal{M}(\emptyset) & \longrightarrow & \mathcal{M}(1) \\ \downarrow & & \downarrow \\ \mathcal{M}(2) & \longrightarrow & \mathcal{M}(12) \end{array} \quad \simeq \quad \begin{array}{ccc} \mathbb{Z}[i]^2 & \longrightarrow & \mathbb{Z}[i] \\ \downarrow & & \downarrow \\ \mathbb{Z}[i] \oplus \mathbb{Z}[i]/(1+i)\mathbb{Z}[i] & \longrightarrow & \mathbb{Z}[i]/2\mathbb{Z}[i] \end{array}$$

The Grothendieck-Tutte polynomial

Let R be a domain, $Q(R)$ its field of fractions. Let $\mathbb{Z}[R\text{-mod}]$ be a ring freely generated as a group by isomorphism classes of f.g. R-modules $[N]$, with product given by $[N][N'] = [N \oplus N']$. Denote by $\vee$ the application of the contravariant functor $\mathrm{Hom}(-, Q(R)/R)$.

Definition

The **Grothendieck-Tutte polynomial** of a matroid $\mathcal{M}$ over a domain R, of rank r on the ground set E is the polynomial:

$$T_{\mathcal{M}}(x, y) = \sum_{A \subseteq [n]} [\mathrm{tor}(A)^\vee](x-1)^{r - \mathrm{rk}(A)}(y-1)^{|A| - \mathrm{rk}(A)}.$$

Theorem (Deletion-Contraction)

Let $\mathcal{M}$ be a matroid over a domain R, of rank r on the ground set E. If $\mathcal{M}(\emptyset)$ is torsion-free and $\mathcal{M}(E) = 0$, then

$$T_{\mathcal{M}}(x, y) = \begin{cases} y T_{\mathcal{M} \setminus i}(x, y) & \text{if } i \text{ is a loop,} \\ x T_{\mathcal{M}/i}(x, y) & \text{if } i \text{ is a coloop,} \\ T_{\mathcal{M} \setminus i}(x, y) + T_{\mathcal{M}/i}(x, y) & \text{otherwise.} \end{cases}$$

The Poset of Torsions

Let $\mathcal{M}$ be a realizable matroid over a domain with a fixed realization ψ. We can associate to $\mathcal{M}$ a (classical) matroid in a natural way. We denote by $\Delta\mathcal{M}$ its independence complex.

Given $A \cup \{b\} \in \Delta\mathcal{M}$, from the definition of matroid over a ring, there is a quotient map $\mathcal{M}(A) \to \mathcal{M}(A \cup \{b\}) \simeq \mathcal{M}(A)/(\psi(b))$ that in a natural way give rise to a surjective map:

$$\pi^\vee_{A,b} : \mathrm{tor}(A \cup \{b\})^\vee \twoheadrightarrow \mathrm{tor}(A)^\vee.$$

Definition

The **poset of torsions** of $\mathcal{M}$ is the set

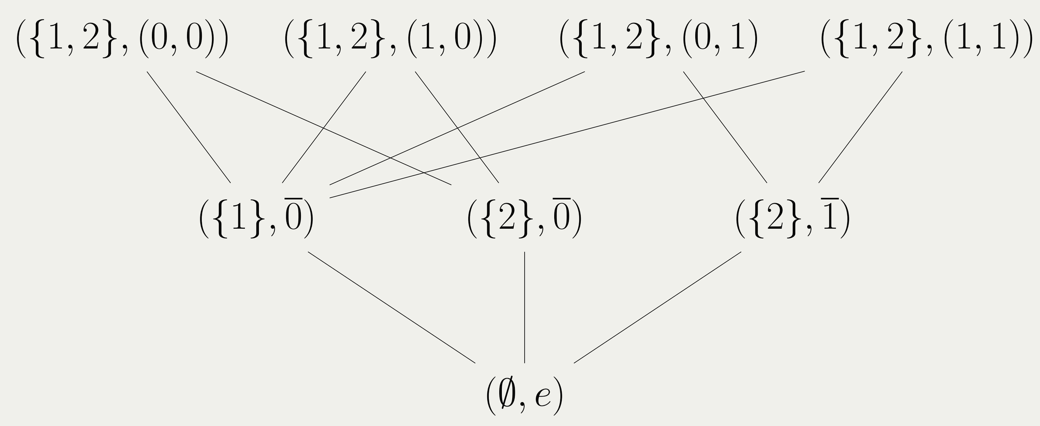
$$\mathrm{Gr}\,\mathcal{M} = \{(A, l) : A \in \Delta\mathcal{M},\ l \in \mathrm{tor}(A)^\vee\},$$

together with the partial order defined by the covering relations $\lessdot$ given as follows: if $(A \cup \{b\}, h), (A, l) \in \mathrm{Gr}\,\mathcal{M}$, then we set

$$(A, l) \lessdot (A \cup \{b\}, h) \overset{\text{def}}{\iff} \pi^\vee_{A,b}(h) = l.$$

Example

The poset of torsions $\mathrm{Gr}\,\mathcal{M}$ of the matroid $\mathcal{M}$ of the previous example is:



Theorem

Let $\mathcal{M}$ be a realizable matroid over a domain R, with a fixed realization. The poset of torsions $\mathrm{Gr}\,\mathcal{M}$ is a disjoint union of simplicial posets, each one isomorphic to $\mathrm{link}(\emptyset, e)$.

Specializations of the Grothendieck-Tutte polynomial

We can associate to a finite simplicial poset L a *Stanley-Reisner ring* (or *face ring*) A_L given by a quotient of $\mathbb{K}[x_a : a \in L]$ by some ideal I_L homogeneous with respect to the grading given by $\deg(x_a) = \mathrm{rk}(a)$.

Now let $\mathbb{F}$ be a *number field* and let R be its ring of integers. We further assume that R is a PID. In these hypothesis, every f.g. torsion R-module N is finite, and $N \simeq N^\vee$.

Let $\mathcal{M}$ be a realizable R-matroid with a fixed realization ψ. In this setting, the poset of torsions of $\mathcal{M}$ is finite.

Definition

Denote by $L = \mathrm{link}(\emptyset, e)$, and let A_L be the face ring of L. The *face module* of $\mathcal{M}$ is the A_L-module

$$A_{\mathcal{M}} = \bigoplus_{t \in \mathrm{tor}(\emptyset)} A_L.$$

Define $\varphi : \mathbb{Z}[R\text{-mod}] \to \mathbb{Z}$ by

$$\varphi([F]) = 1 \quad \text{for every free module } F,$$
$$\varphi([N]) = |N| \quad \text{for every torsion module } N.$$

We can specialize the Grothendieck-Tutte polynomial, using the homomorphism φ, to obtain a formula for the Hilbert series of $A_{\mathcal{M}}$.

Theorem

$$H(A_{\mathcal{M}}, t) = \frac{t^r}{(1-t)^r}\varphi(T_{\mathcal{M}}(1/t, 1))$$

Example

The face module of the matroid $\mathcal{M}$ in the previous examples is:

$$A_{\mathcal{M}} \simeq \frac{\mathbb{K}[x_a, x_{b_0}, x_{b_1}, x_{c_0}, x_{c_1}, x_{d_0}, x_{d_1}]}{\left(\begin{array}{l} x_a x_{b_i} - (x_{c_i} + x_{d_i}), x_{b_0}x_{b_1}, \\ x_{c_i}x_{d_j}, x_{c_0}x_{c_1}, x_{d_0}x_{d_1}, \\ x_{b_i}x_{c_{\bar{i}}}, x_{b_i}x_{d_{\bar{i}}} \end{array} : \begin{array}{l} i, j \in \{0, 1\} \\ \bar{i} = 1 - i \end{array} \right)}$$

In particular, we have:

$$H(A_{\mathcal{M}}, t) = \frac{1 + t + 2t^2}{(1-t)^2} = \frac{t^2}{(1-t)^2}\varphi\Big(T_{\mathcal{M}}(1/t, 1)\Big).$$

References

[1] Alessio Borzì and Ivan Martino. Set of independencies and tutte polynomial of matroids over a domain. *arXiv preprint arXiv:1909.00332*, 2019.

[2] Alex Fink and Luca Moci. Matroids over a ring. *Journal of the European Mathematical Society*, 18(4):681–731, 2016.