Realizability of quotients of matroids

Alessio Borzì

MPI MiS Leipzig

Università di Catania

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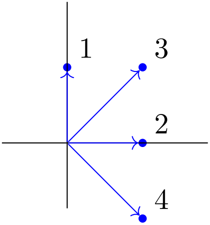

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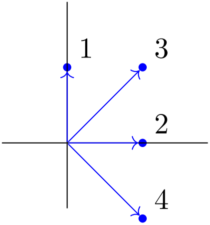

$$U = \text{rowspace}(A) \subseteq \mathbb{C}^4$$

$$\dim U = 2$$

$$E = \{1, 2, 3, 4\}$$

$$\mathcal{I} = \{I \subseteq E : \text{ the vectors in } I \text{ are l.i. } \}$$

$$= \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \dots, \{3, 4\}\}$$

Matroid definition

Definition

A matroid M is a pair $(E, \mathcal{I})$ where:

- E *is a set called ground set,*
- $\mathcal{I}$ *is a family of subsets of* E*, called independent subsets, s.t.*
  1. $\emptyset \in \mathcal{I}$
  2. $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$
  3. $A, B \in \mathcal{I}$, $|A| < |B| \Rightarrow \exists b \in B \setminus A : A \cup \{b\} \in \mathcal{I}$.

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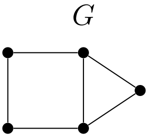
If a matroid M arises from a set of vectors in a K-vector space, it is realizable over K.

Matroids from Graphs

Let $G = (V, E)$ be a graph



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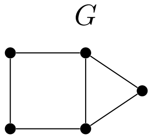

$$\mathcal{I} = \{A \subseteq E : A \text{ is a subtree } \}$$

$M(G) = (E, \mathcal{I})$ is a matroid, called the cycle matroid of G. Matroids arising in this way are called graphic matroids.

Realizability

Graphic matroids are realizable over every field.

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Not all matroids are realizable over every field!

Example

$U_{2,4} = (E, \mathcal{I})$ where

$$E = \{1, 2, 3, 4\}$$
$$\mathcal{I} = \{A \subseteq E : |A| \leq 2\}$$

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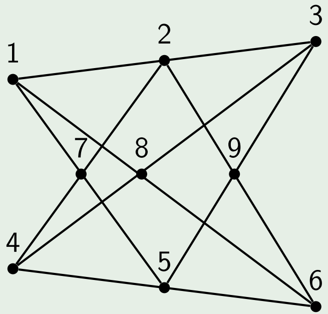
$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & x \end{bmatrix}$$

Non-Pappus matroid

Example (Non-Pappus matroid)



P

Not realizable over any field.

Deletion and Contraction

Let $M = (E, \mathcal{I})$ be a matroid and let $e \in E$

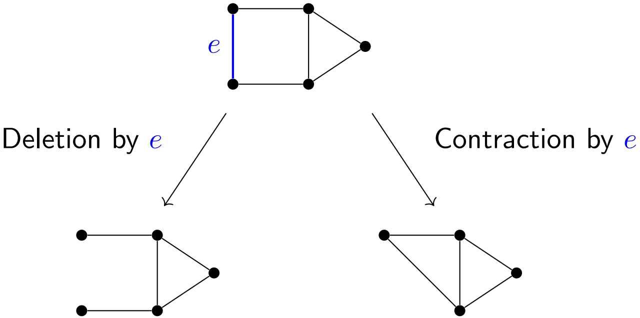
- **Deletion** $M \setminus e$: $\mathcal{I}(M \setminus e) = \{I : I \in \mathcal{I}, e \notin I\}$
- **Contraction** M/e: $\mathcal{I}(M/e) = \{I \setminus e : I \in \mathcal{I}, e \in I\}$

Both $M \setminus e$ and M/e have ground set $E \setminus e$.

Deletion and Contraction: realizable matroids

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{c} \begin{array}{ccccc} {\color{blue}e} & & & & \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix} \end{array} \arrow[dl, "\text{Deletion by } {\color{blue}e}"'] \arrow[dr, "\text{Contraction by } {\color{blue}e}"] & \\
\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} & & \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}
\end{tikzcd}$$

Deletion and Contraction: graphic matroids



Rank and flats

Let M be a matroid on E.

The rank $r(A)$ of $A \subseteq E$ is the cardinality of a maximal independent set contained in A.

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The rank $r(A)$ of $A \subseteq E$ is the cardinality of a maximal independent set contained in A.

The rank of M is the rank of its ground set E.

$F \subseteq E$ is a flat if $r(F \cup x) > r(F)$ for all $x \in E \setminus F$.

Matroid quotients

Definition

Let M and N be two matroids on the same ground set E. Then, N is a matroid quotient of M if every flat of N is a flat of M.

$$M \twoheadrightarrow N$$

Example of matroids quotient

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \longrightarrow\!\!\!\!\!\twoheadrightarrow \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Example of matroids quotient

$$
\begin{tikzcd}
& \begin{array}{l} \;\;e \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix} \end{array} \arrow[dl, "\text{Deletion by } e"'] \arrow[dr, "\text{Contraction by } e"] & \\
\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix} \arrow[rr, two heads] & & \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}
\end{tikzcd}
$$

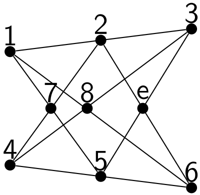
Realizable matroid quotients

A matroid quotient $M \twoheadrightarrow N$ is realizable over K if:

- M is realized by $V \subseteq K^n$,
- N is realized by $U \subseteq K^n$,
- $U \subseteq V$.

$$\begin{tikzcd} N \arrow[d, dash] & M \arrow[l, twoheadrightarrow] \arrow[d, dash] \\ U \arrow[r, phantom, "\subseteq"] & V \end{tikzcd}$$

Non realizable matroid quotient



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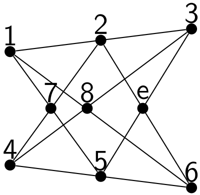

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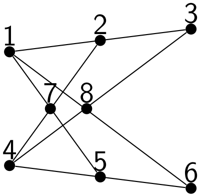
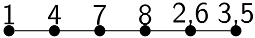
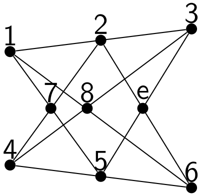


$P \setminus e$



P/e

Non realizable matroid quotient



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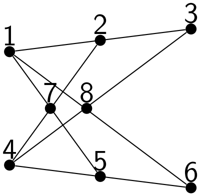
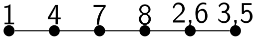


$P \setminus e$

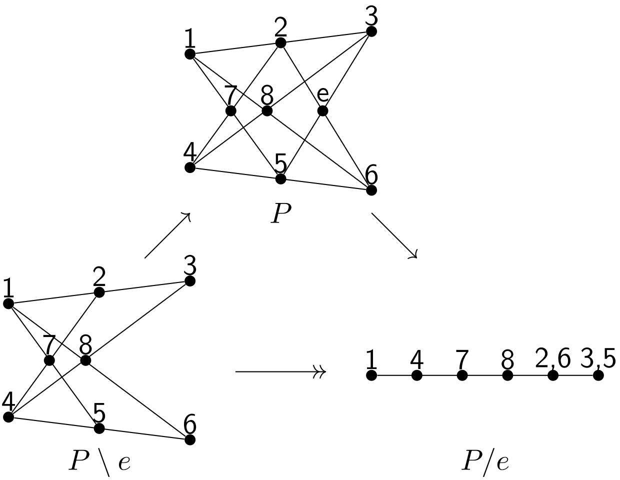
$\longrightarrow\!\!\!\!\rightarrow$



P/e

$P \setminus e$ and P/e are realizable, but $P \setminus e \twoheadrightarrow P/e$ is not

Non realizable matroid quotient



$P \setminus e$ and P/e are realizable, but $P \setminus e \twoheadrightarrow P/e$ is not

Let M be a matroid on E. An extension of M is a matroid H on $E \cup S$ such that $M = H \setminus S$.

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Theorem (Higgs)

Every matroid quotient $M \twoheadrightarrow N$ can be factored into an extension, followed by a contraction:

$$\begin{tikzcd} & H \arrow[dr, "/S"] & \\ M \arrow[ur, "+S"] \arrow[rr, two heads] & & N \end{tikzcd}$$

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A matroid H with the property above is a major of $M \twoheadrightarrow N$.

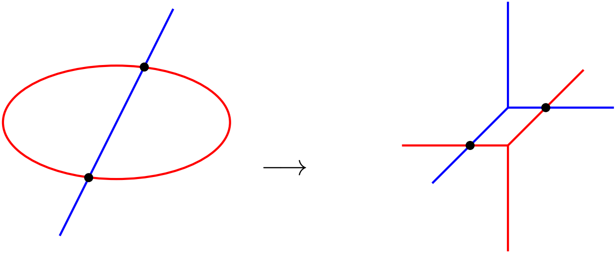
The major constructed in Higgs theorem is called Higgs major.

Theorem (B. 2024)

Let K be an infinite field. TFAE:

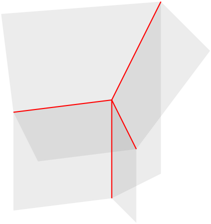
1. $M \twoheadrightarrow N$ *is realizable over* K,
2. *the Higgs major* H *of* $M \twoheadrightarrow N$ *is realizable over* K.

Tropical geometry



$V(I)$ $\quad$ $V(J)$ $\qquad \longrightarrow \qquad$ $\operatorname{trop}(V(I))$ $\quad$ $\operatorname{trop}(V(J))$

Relative realizability problem

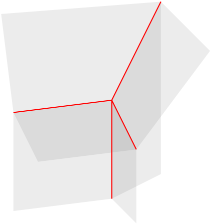


$\mathcal{Y} \subseteq \mathcal{X}$

Problem (Relative realizability problem in tropical geometry)

Given a pair of tropical varieties $\mathcal{Y} \subseteq \mathcal{X}$ and an algebraic variety X tropicalizing to $\mathcal{X}$, does there exist a subvariety $Y \subseteq X$ tropicalizing to $\mathcal{Y}$?

Relative realizability problem



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Given a pair of tropical varieties $\mathcal{Y} \subseteq \mathcal{X}$ and an algebraic variety X tropicalizing to $\mathcal{X}$, does there exist a subvariety $Y \subseteq X$ tropicalizing to $\mathcal{Y}$?

The answer might be negative even when $\mathcal{X}$ and $\mathcal{Y}$ are linear, which is the case we will focus on.

Tropical linear spaces

$I \subseteq K[x_0, \ldots, x_n]$ homogeneous linear ideal.

I_1 is a K-vector space, with underlying matroid $M(I_1)$.

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The tropical linear space $\operatorname{trop}(V(I))$ depends just on $M(I_1)$.

Let $\operatorname{trop}(M)$ be the tropical linear space associated to M.

Relative realizability for tropical linear spaces

Let $\mathcal{X} = \mathrm{trop}(M_1)$ and $\mathcal{Y} = \mathrm{trop}(M_2)$.

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TFAE:

- $\mathcal{Y} \subseteq \mathcal{X}$ *is realizable over* K*,*
- $M_1 \twoheadrightarrow M_2$ *is realizable over* K*.*

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TFAE:

- $\mathcal{Y} \subseteq \mathcal{X}$ *is realizable over* K,
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Corollary

If K *is infinite, TFAE:*

- $\mathcal{Y} \subseteq \mathcal{X}$ *is realizable over* K,
- *the Higgs major* H *of* $M_1 \twoheadrightarrow M_2$ *is realizable over* K.

Lamboglia's example

$L \subseteq \mathbb{P}^5(\mathbb{C})$ plane

Lamboglia's example

$$\begin{tikzcd} L \arrow[d] \\ \mathrm{trop}(L) \end{tikzcd}$$

standard tropical plane

Lamboglia's example

$$\begin{array}{ccc} & & L \\ & & \big\downarrow \\ \Gamma & \subseteq & \mathrm{trop}(L) \end{array}$$

realizable tropical line

Lamboglia's example

$$\begin{array}{ccc} \ell & \not\subseteq & L \\ \big\downarrow & & \big\downarrow \\ \Gamma & \subseteq & \mathrm{trop}(L) \end{array}$$

No line ℓ tropicalizing to Γ is contained in L!

Lamboglia's example

$$M(\ell) \twoheadleftarrow\!\!\longrightarrow M(L)$$

Lamboglia's example

$$
\begin{tikzcd}
& H \arrow[dl] & \\
M(\ell) & & M(L) \arrow[ul] \arrow[ll, two heads]
\end{tikzcd}
$$

Lamboglia's example

$$\begin{tikzcd}
 & H \arrow[dl] & \\
M(\ell) & & M(L) \arrow[ll, two heads] \arrow[ul]
\end{tikzcd}$$

L is a realization of $M(L)$ that does not extend to H.

Vita quieta,
mente lieta,
moderata dieta.

Quiet life,
happy mind,
moderate diet.

Thank you for your attention!

Alessio Borzì.
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