Tropical moduli spaces and toric embeddings

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$\overline{M}_{0,n}$ seminar

$$M_{0,n} \to \overline{M}_{0,n}$$

as a tropical compactification

Let $T^n = (\mathbb{C}^*)^n$ be the n-dimensional *algebraic torus*

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Proposition

1. $\overline{Y}$ *is complete* $\iff \operatorname{trop}(Y) \subseteq |\Sigma|$.

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Proposition

1. *$\overline{Y}$ is complete* $\iff \mathrm{trop}(Y) \subseteq |\Sigma|$.
2. *If $\overline{Y}$ is complete, then*

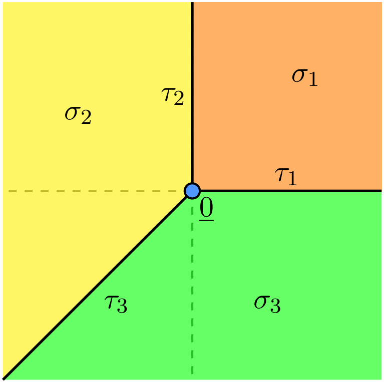
   $Y \cap \mathcal{O}_\sigma$ *is pure of dimension* $\dim(Y) - \dim(\sigma)$, $\forall \sigma \in \Sigma$ $\iff \mathrm{trop}(Y) = |\Sigma|$

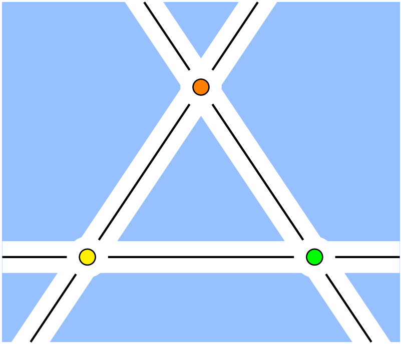
Tropical compactification

Definition (Tropical Compactification)

A ***tropical compactification*** *of* $Y \subseteq T^n$ *is its closure* $\overline{Y}$ *in a toric variety with* X_Σ *with* $|\Sigma| = \mathrm{trop}(Y)$.

Example



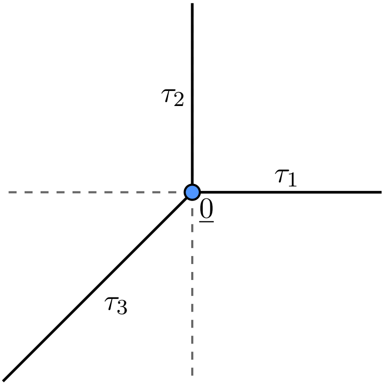


Σ $\qquad$ $X_\Sigma = \mathbb{P}^2$

Three coordinate points: (0:0:1) (1:0:0) (0:1:0)

The light blue part is T^2

Example



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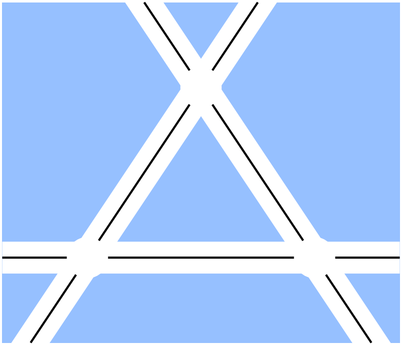

$X_\Sigma = \mathbb{P}^2 \setminus 3$ points

Example

$$Y = V(x + y + 1) \subseteq T^2$$

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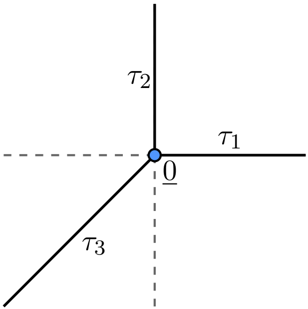

$\mathrm{trop}(Y)$

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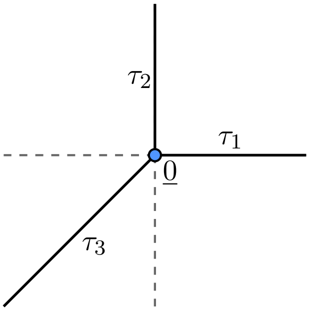
$$Y = V(x+y+1) \subseteq T^2 \hookrightarrow \mathbb{P}^2 \setminus 3 \text{ points}$$



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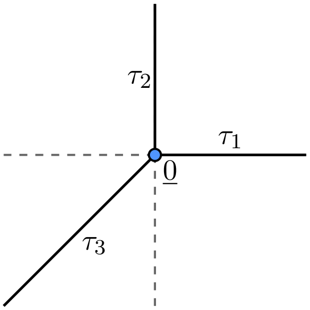

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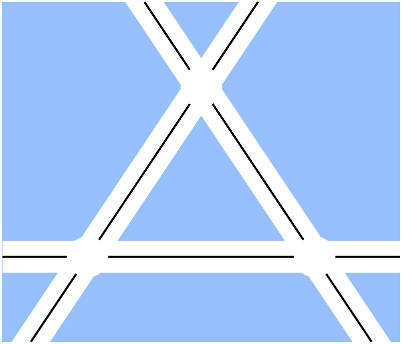

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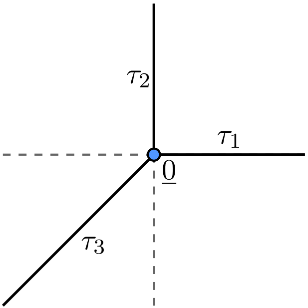

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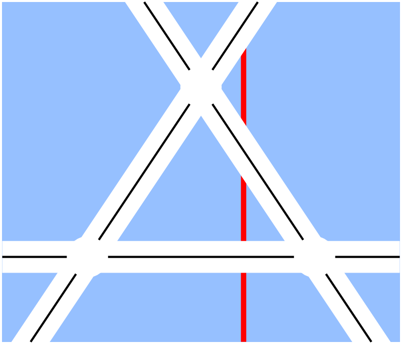

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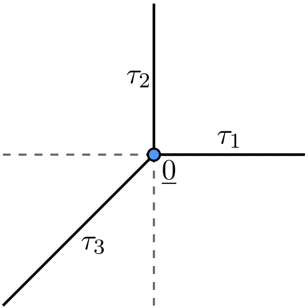

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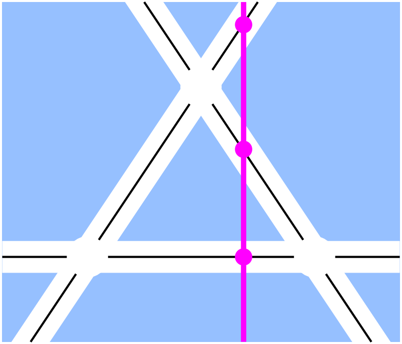

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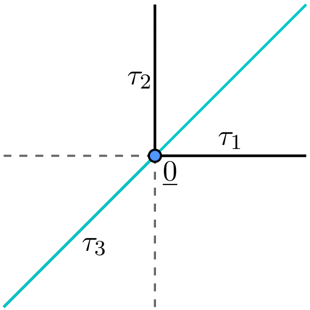

trop(Y)

Example 2

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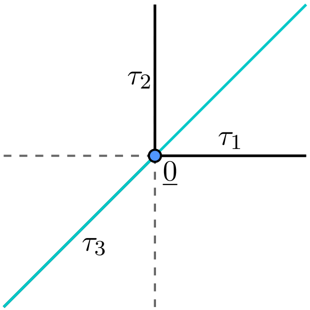

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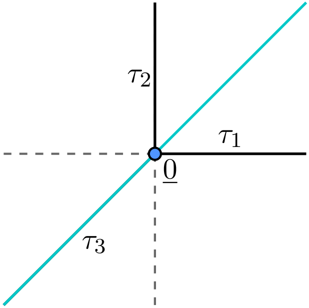

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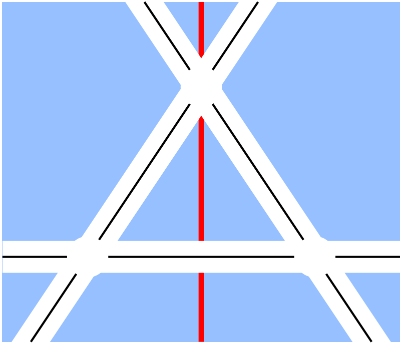

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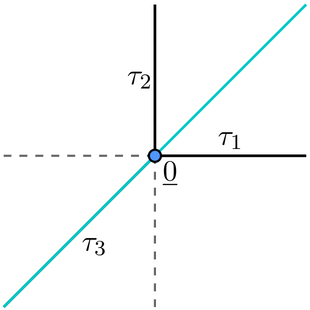

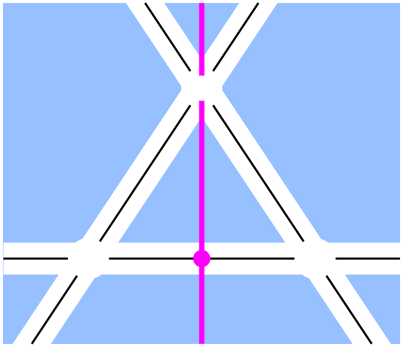

$\text{trop}(Y) \neq |\Sigma|$

$\overline{Y} \subseteq X_\Sigma$

$\overline{Y} \simeq \mathbb{P}^1 \setminus \text{ point}$

$$M_{0,n} = (\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3} \setminus \text{ diagonals}$$

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We have realized $M_{0,n}$ as the complement of $\binom{n-1}{2}$ hyperplanes, with equations given by the columns of the matrix:

$$B = \left[\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & \dots & 1 & 1 & \dots & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots & -1 & 0 & \dots & 1 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots & 0 & -1 & \dots & -1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & \dots & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right] \in \mathbb{C}^{(n-2), \binom{n-1}{2}}$$

$$\underbrace{\qquad\qquad}_{\text{identity matrix}} \qquad \underbrace{\qquad\qquad\qquad}_{\text{columns given by } e_i - e_j}$$

This gives us a closed embedding

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Viewed inside $\mathbb{P}^{\binom{n-1}{2}-1}$ it is defined by the homogeneous ideal

$$I_{0,n} = \langle z_{ij} - z_{1j} + z_{1i} : 2 \leq i, j \leq n-1 \rangle \subseteq \mathbb{C}[z_{ij}]$$

trop($M_{0,n}$) and phylogenetic trees

Let Δ be a fan with $|\Delta| = \mathrm{trop}(M_{0,n})$ with the coarser fan structure. This fan is the space of *phylogenetic trees*.

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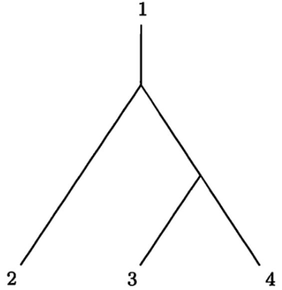
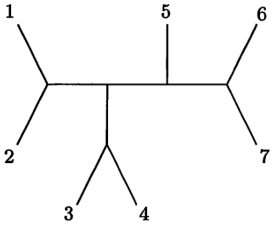
Definition

A ***phylogenetic tree*** *is a tree with no vertices of degree 2.*





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Definition

The vector $(d_{ij}) \in \mathbb{R}^{\binom{n}{2}}$ is a ***tree distance****. The* ***space of phylogenetic trees*** *is the set Δ of all tree distances.*

For every leaf i, adding the vector $\sum_{j \neq i} e_{ij}$ to a tree distance (d_{ij}) corresponds to adding 1 to the length of the pendant edge containing i.

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Therefore, we can view Δ as a subset of $\mathbb{R}^{\binom{n}{2}}/L \simeq \mathbb{R}^{\binom{n-1}{2}}$.

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Therefore, we can view Δ as a subset of $\mathbb{R}^{\binom{n}{2}}/L \simeq \mathbb{R}^{\binom{n-1}{2}}$. The combinatorial types of the phylogenetic trees give Δ the structure of a fan in a natural way.

Theorem

The closure of $M_{0,n}$ in the corresponding toric variety X_Δ is isomorphic to the Deligne-Mumford compactification $\overline{M}_{0,n}$.

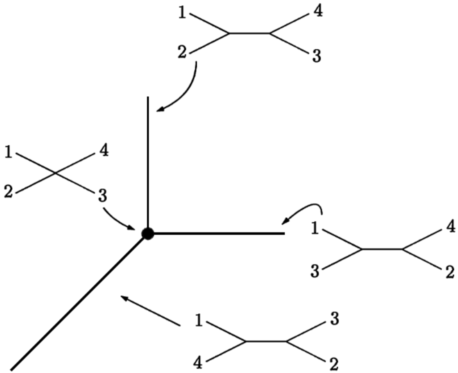
Example: $M_{0,4}$

Let $n = 4$ and consider $M_{0,4} \simeq \mathbb{P}^1 \setminus \{0, 1, \infty\}$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$I_{0,4} = \langle -z_{12} + z_{13} + z_{23} \rangle$$

We can think it as being $M_{0,4} = V(x+y+1)$. The space Δ of phylogenetic trees is then the tropicalization of a general line:



Example: $M_{0,5}$

Let $n = 5$ and consider $M_{0,5}$

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

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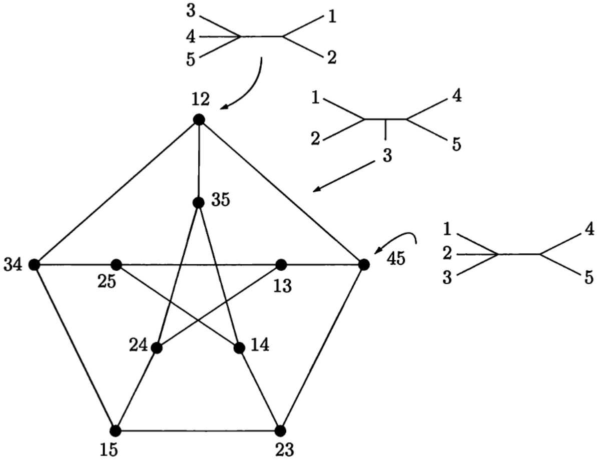
$$B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

The equations of $M_{0,5}$ are given by the row of the kernel of B

$$\ker B = \begin{array}{c} \begin{array}{cccccc} z_{12} & z_{13} & z_{14} & z_{23} & z_{24} & z_{34} \end{array} \\ \begin{bmatrix} -1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$I_{0,5} = \langle -z_{12} + z_{13} + z_{23}, -z_{12} + z_{14} + z_{24}, -z_{13} + z_{14} + z_{34} \rangle$$

The space of phylogenetic trees Δ is a fan of dimension 2 that can be thought as the cone over the *Petersen graph*:



Intersection theory of $\overline{M}_{0,n}$

Theorem

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Where $A^*(X_\Delta)$ can be explicitly described as follows. Suppose Δ has s rays and let D_i be the divisor of X_Δ corresponding to the i-th ray. Define

$$\mathrm{SR}(\Delta) = \left\langle \prod_{i\in\sigma} D_i : \sigma \notin \Delta \right\rangle, \quad L_\Delta = \left\langle \sum_{j=1}^{s} V_{ij} D_j : 1 \leq i \leq n \right\rangle,$$

where $V = (V_{ij})$ is the matrix with columns the first lattice points of the rays of Δ. We have

$$A^*(X_\Delta) = \mathbb{Z}[D_1, \ldots, D_s]/(\mathrm{SR}(\Delta) + L_\Delta)$$

Thank you for your attention!