Realizability of quotients of matroids

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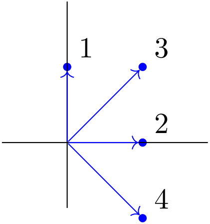
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \in \mathbb{C}^{2,4}$$

$$A = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \end{array} \in \mathbb{C}^{2,4}$$



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$$\dim U = 2$$

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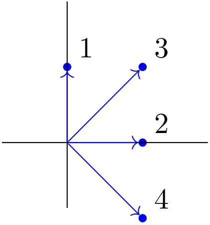

$$U = \operatorname{rowspace}(A) \subseteq \mathbb{C}^4$$

$$\dim U = 2$$

$$E = \{1, 2, 3, 4\}$$

$$\mathcal{I} = \{I \subseteq E : \text{ the vectors in } I \text{ are l.i. } \}$$

Matroid definition

Definition

A matroid M *is a pair* $(E, \mathcal{I})$ *where:*

- E *is a set called ground set,*
- $\mathcal{I}$ *is a family of subsets of* E*, called independent subsets, s.t.*
  1. $\emptyset \in \mathcal{I}$
  2. $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$
  3. $A, B \in \mathcal{I}, |A| < |B| \Rightarrow \exists b \in B \setminus A : A \cup \{b\} \in \mathcal{I}.$

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If a matroid M arises from a set of vectors in a K-vector space, it is realizable over K.

Non-Pappus matroid

Not all matroids are realizable!

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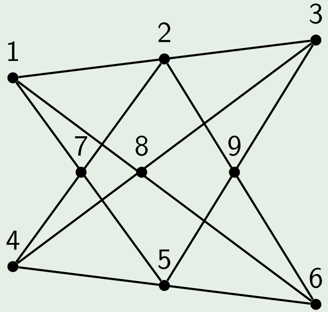
Example (Non-Pappus matroid)



P

Not realizable over any field.

Rank and flats

Let M be a matroid on E.

The rank $r(A)$ of $A \subseteq E$ is the cardinality of a maximal independent set contained in A.

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The rank of M is the rank of its ground set E.

$F \subseteq E$ is a flat if $r(F \cup x) > r(F)$ for all $x \in E \setminus F$.

Matroid quotients

Definition

Let M and N be two matroids on the same ground set E. Then, N is a matroid quotient of M if every flat of N is a flat of M.

$$M \twoheadrightarrow N$$

Realizable matroid quotients

A matroid quotient $M \twoheadrightarrow N$ is realizable over K if:

- M is realized by $V \subseteq K^n$,
- N is realized by $U \subseteq K^n$,
- $U \subseteq V$.

$$\begin{tikzcd}
N \arrow[d, dash] & M \arrow[l, two heads] \arrow[d, dash] \\
U \arrow[r, phantom, "\subseteq"] & V
\end{tikzcd}$$

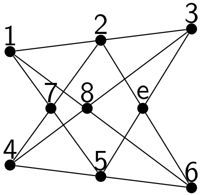
Deletion and Contraction

Let $M = (E, \mathcal{I})$ be a matroid and let $e \in E$

- **Deletion** $M \setminus e$: $\mathcal{I}(M \setminus e) = \{I : I \in \mathcal{I}, e \notin I\}$
- **Contraction** M/e: $\mathcal{I}(M/e) = \{I \setminus e : I \in \mathcal{I}, e \in I\}$

Both $M \setminus e$ and M/e have ground set $E \setminus e$.

Non realizable matroid quotient



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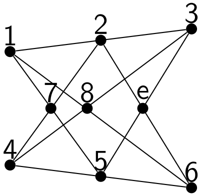

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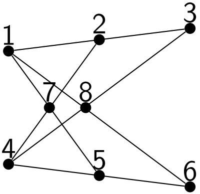
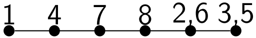
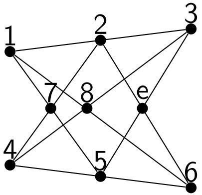


$P \setminus e$



P/e

Non realizable matroid quotient



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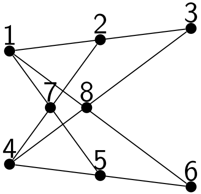
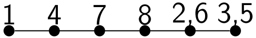


$P \setminus e$

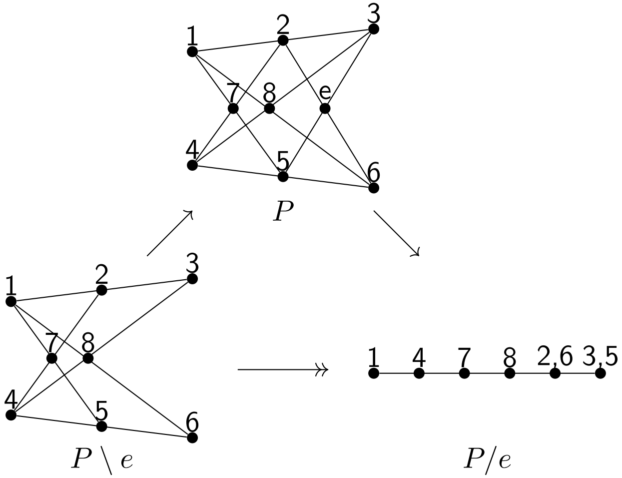
$\twoheadrightarrow$



P/e

$P \setminus e$ and P/e are realizable, but $P \setminus e \twoheadrightarrow P/e$ is not

Non realizable matroid quotient



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Let M be a matroid on E. An extension of M is a matroid H on $E \cup S$ such that $M = H \setminus S$.

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Theorem (Higgs)

Every matroid quotient $M \twoheadrightarrow N$ can be factored into an extension, followed by a contraction:

$$\begin{tikzcd}
 & H \arrow[dr, "/S"] & \\
M \arrow[ur, "+S"] \arrow[rr, two heads] & & N
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A matroid H with the property above is a major of $M \twoheadrightarrow N$.

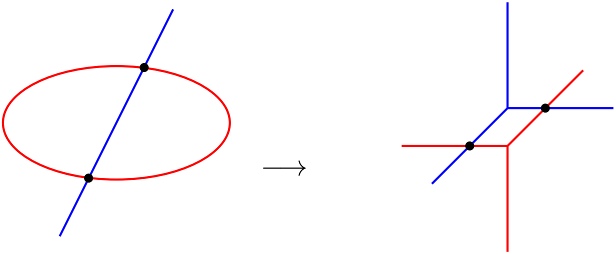
The major constructed in Higgs theorem is called Higgs major.

Theorem (B. 2024)

Let K be an infinite field. TFAE:

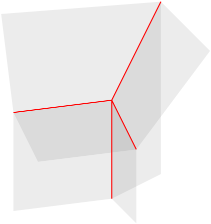
1. $M \twoheadrightarrow N$ *is realizable over* K,
2. *the Higgs major* H *of* $M \twoheadrightarrow N$ *is realizable over* K.

Tropical geometry



$V(I)$ $V(J)$ $\longrightarrow$ $\operatorname{trop}(V(I))$ $\operatorname{trop}(V(J))$

Relative realizability problem

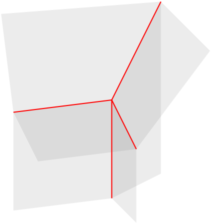


$\mathcal{Y} \subseteq \mathcal{X}$

Problem (Relative realizability problem in tropical geometry)

Given a pair of tropical varieties $\mathcal{Y} \subseteq \mathcal{X}$ *and an algebraic variety* X *tropicalizing to* $\mathcal{X}$*, does there exist a subvariety* $Y \subseteq X$ *tropicalizing to* $\mathcal{Y}$*?*

Relative realizability problem



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Problem (Relative realizability problem in tropical geometry)

Given a pair of tropical varieties $\mathcal{Y} \subseteq \mathcal{X}$ and an algebraic variety X tropicalizing to $\mathcal{X}$, does there exist a subvariety $Y \subseteq X$ tropicalizing to $\mathcal{Y}$?

The answer might be negative even when $\mathcal{X}$ and $\mathcal{Y}$ are linear, which is the case we will focus on.

Tropical linear spaces

$I \subseteq K[x_0, \ldots, x_n]$ homogeneous linear ideal.

I_1 is a K-vector space, with underlying matroid $M(I_1)$.

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The tropical linear space $\mathrm{trop}(V(I))$ depends just on $M(I_1)$.

Let $\mathrm{trop}(M)$ be the tropical linear space associated to M.

Relative realizability for tropical linear spaces

Let $\mathcal{X} = \mathrm{trop}(M_1)$ and $\mathcal{Y} = \mathrm{trop}(M_2)$.

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Proposition

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TFAE:

- $\mathcal{Y} \subseteq \mathcal{X}$ *is realizable over* K*,*
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TFAE:

- $\mathcal{Y} \subseteq \mathcal{X}$ *is realizable over* K*,*
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Corollary

If K *is infinite, TFAE:*

- $\mathcal{Y} \subseteq \mathcal{X}$ *is realizable over* K*,*
- *the Higgs major* H *of* $M_1 \twoheadrightarrow M_2$ *is realizable over* K*.*

Lamboglia's example

$L \subseteq \mathbb{P}^5(\mathbb{C})$ plane

Lamboglia's example

$$\begin{tikzcd}
L \arrow[d] \\
\mathrm{trop}(L)
\end{tikzcd}$$

standard tropical plane

Lamboglia's example

$$\begin{tikzcd}
 & L \arrow[d] \\
\Gamma \arrow[r, phantom, "\subseteq"] & \mathrm{trop}(L)
\end{tikzcd}$$

realizable tropical line

Lamboglia's example

$$\begin{tikzcd}
\ell \arrow[r, phantom, "\not\subseteq"] \arrow[d] & L \arrow[d] \\
\Gamma \arrow[r, phantom, "\subseteq"] & \mathrm{trop}(L)
\end{tikzcd}$$

No line ℓ tropicalizing to Γ is contained in L!

Lamboglia's example

$$\begin{tikzcd}M(\ell) & M(L) \arrow[l, two heads]\end{tikzcd}$$

Lamboglia's example

$$\begin{tikzcd}
& H \arrow[dl] & \\
M(\ell) & & M(L) \arrow[ll, two heads] \arrow[ul]
\end{tikzcd}$$

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L is a realization of $M(L)$ that does not extend to H.

Thank you for your attention!