Cyclotomic Numerical Semigroups and Graded Algebras

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York Semigroup Seminar

25 October 2023

Part I: Cyclotomic Numerical Semigroups

Part II: Algebraic point of view

Numerical semigroups

A **numerical semigroup** is a subset $S \subseteq \mathbb{N}$ such that

- $0 \in S$;
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Every numerical semigroup is finitely generated, and has a unique minimal set of generators.

So every numerical semigroup is of the form $S = \langle n_1, \ldots, n_e \rangle$ with $\gcd(n_1, \ldots, n_e) = 1$.

Semigroup polynomial

Definition

The **semigroup polynomial** *of S is*

$$P_S(x) = 1 + (x - 1) \sum_{g \in \mathbb{N} \setminus S} x^g$$

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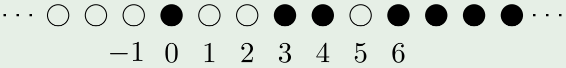
Example

$$S = \langle 3, 4\rangle$$



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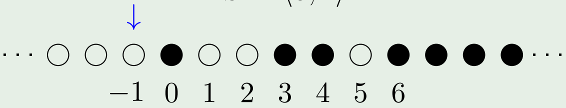

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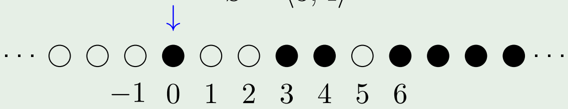

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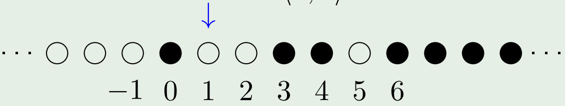

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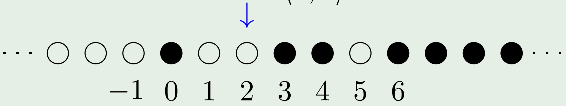

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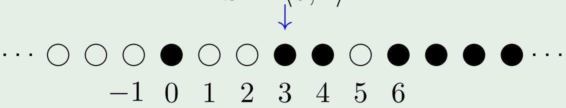

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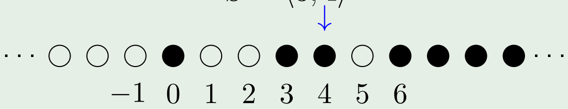

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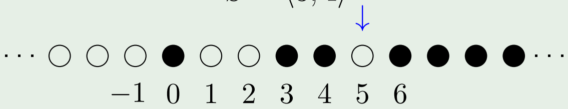

$$P_S(x) = 1 - x + x^3 - x^5$$

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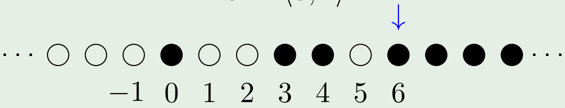

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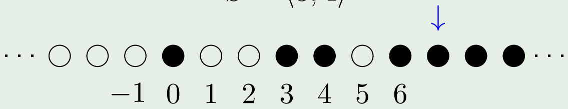

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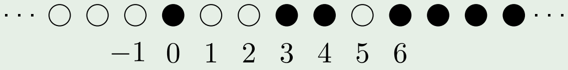

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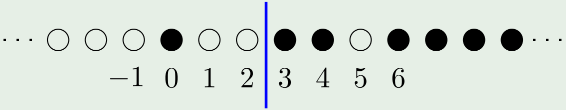

$$\frac{F(S)}{2} = 2.5$$

$$F(S) = 5$$

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Theorem

S is symmetric if and only if $P_S(x)$ is palindromic.

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$$S = \langle 3, 4 \rangle$$

$$P_S(x) = 1x^0 + (-1)x^1 + 0x^2 + 1x^3 + 0x^4 + (-1)x^5 + 1x^6$$

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$$\begin{aligned} P_S(x) &= 1 - x + x^3 - x^5 + x^6 \\ &= (1 - x + x^2)\,(1 - x^2 + x^4) \end{aligned}$$

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Complete intersection numerical semigroups

Let S, S_1, S_2 be n. s. and let $a_1 \in S_2$ and $a_2 \in S_1$ such that they are coprime and not minimal generators of their semigroups.

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- $S = a_1 S_1 + a_2 S_2$,
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Definition

S is a **complete intersection** *if*

- *$S = \mathbb{N}$, or*
- *S is the gluing of two complete intersection numerical semigroups.*

Complete intersection numerical semigroups

Example

$$S = \langle 3, 4 \rangle = 3\mathbb{N} + 4\mathbb{N}$$

$$S = \langle 6, 7, 8 \rangle = 7\mathbb{N} + 2\langle 3, 4 \rangle$$

complete intersection

cyclotomic

symmetric

complete intersection $\Longrightarrow$ cyclotomic $\Longrightarrow$ symmetric

$$\begin{array}{c}\text{complete}\\\text{intersection}\end{array} \implies \text{cyclotomic} \;\begin{array}{c}\implies\\ \not\Longleftarrow\end{array}\; \text{symmetric}$$

Herrera-Poyatos, Moree and independently Sawhney, Stoner:

Theorem

$S_k = \langle k, k+1, \ldots, 2k-2 \rangle$ is symmetric but not cyclotomic for every $k \geq 5$.

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Conjecture (Ciolan, García-Sánchez, Moree 2016)

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Theorem (Herzog 1969)

If $e(S) \leq 3$ then S is symmetric iff it is a complete intersection.

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Theorem (B., Herrera-Poyatos, Moree 2021)

If $P_S(x)$ has at most 2 irreducible factors then S is cyclotomic iff it is a complete intersection.

An algebraic point of view

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$$k[S] = k[t^s : s \in S]$$

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$$H(k[S], x) = \frac{\mathcal{K}(k[S], x)}{(1 - x^{n_1}) \dots (1 - x^{n_e})}$$

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Definition

A graded algebra R is **cyclotomic** *if $N_R(x)$ is a product of cyclotomic polynomials.*

Gorenstein algebras

Theorem (Kunz 1970)

$k[S]$ is Gorenstein iff S is symmetric.

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Corollary

If R is a cyclotomic Cohen-Macaulay graded domain then R is Gorenstein.

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Corollary

If R is a complete intersection, it is cyclotomic.

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$$\begin{matrix}\text{complete}\\ \text{intersection}\end{matrix} \implies \text{cyclotomic} \overset{\left(\substack{\text{Cohen-Macaulay}\\ \text{domains}}\right)}{\implies} \text{Gorenstein}$$

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Example (Stanley)

$R = k[x, y]/(x^3, xy, y^2)$ with $\deg(x) = \deg(y) = 1$. We have

$$H(R, t) = \frac{1 - 2t^2 + t^4}{(1-t)^2} = (1+t)^2.$$

R is cyclotomic, but not a complete intersection.

Koszul algebras

Definition

A graded algebra $R \simeq k[x_1, \ldots, x_e]/I$ *with* $\deg(x_i) = 1$ *is* **Koszul** *if the minimal free resolution of* k *as an* R*-module is linear (i.e.* $\beta_{i,j}^R(k) = 0$ *whenever* $i \neq j$*).*

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Theorem (B., D'Alì 2021)

A Koszul algebra R is cyclotomic iff it is a complete intersection.

Thank you for your attention!